



## Vibration analysis of functionally graded cylindrical shells with different boundary conditions subjected to thermal loads

M. Talebitooti\*, M. Ghasemi, S. M. Hosseini

Department of Mechanical Engineering, Qom University of Technology, Qom, 1519-37195, Iran

### Article info:

Received: 04/11/2015  
Accepted: 01/02/2017  
Online: 03/03/2017

### Keywords:

Functionally graded materials,  
Cylindrical shells,  
Natural frequency,  
First-order shear deformation theory (FSDT),  
Thermal load.

### Abstract

In the present work, study of the vibration of a functionally graded (FG) cylindrical shell made up of stainless steel, zirconia, and nickel is presented. Free vibration analysis is presented for FG cylindrical shells with simply supported-simply supported and clamped-clamped boundary condition based on temperature independent material properties. The equations of motion are derived by Hamilton's principle. Material properties assume to be graded in the thickness direction according to a simple power law distribution in terms of the volume fraction of the constituents. Effects of boundary conditions and volume fractions (power law exponent) on the natural frequencies of the FG cylindrical shell are studied. Frequency characteristics of the FG shell are found to be similar to those of isotropic cylindrical shells. Furthermore, natural frequencies of these shells are observed to be dependent on the constituent volume fractions and boundary conditions. Strain displacement relations from Love's and first-order shear deformation theories are employed. Galerkin method is used to derive the governing equations for clamped boundary conditions. Further, analytical results are validated with those reported in the literature and excellent agreement is observed. Finally, in order to investigate the effects of the temperature gradient, functionally graded materials cylindrical shell with high temperature specified on the inner surface and outer surface at ambient temperature, 1D heat conduction equation along the thickness of the shell is applied and the results are reported.

### Nomenclature

$A_{ij}$	extensional stiffness	$Q$	resultant shear force
$B_{ij}$	coupling stiffness	$R$	radius of the shell
$D_{ij}$	bending stiffness	$T$	kinetic energy
$E$	modulus of Young's elasticity	$U$	strain energy
$H$	thickness of shell	$W$	work carried on the shell
$K$	Shear correction factor	<b>Greek letters</b>	
$L$	length of shell	$\varepsilon$	middle surface strain
$M$	resultant moment	$\nu$	Poisson's ratio
$N$	resultant force	$\rho$	mass density
$N^T$	thermal load	$\omega$	natural frequency in rad/s
$n$	circumferential wave number		

\*Corresponding author  
email address: [talebi@qut.ac.ir](mailto:talebi@qut.ac.ir)

## 1. Introduction

In the recent years, functionally gradient materials (FGMs) have gained considerable attention in the high temperature environment applications. Many familiar FGMs are compositionally graded from a refractory ceramic to a metal so that it can incorporate incompatible functions such as the heat, wear and oxidation resistance of ceramics and the high toughness, high strength, machine-ability and bonding capability of metals without severe internal thermal stress. The concept of FGMs was first introduced in 1984 by a group of materials scientists in Japan [1, 2], as a means of preparing thermal barrier materials. Since then FGMs have attracted much interest as heat-shielding materials. FGMs are made by combining different materials using powder metallurgy methods [3]. They possess variations in constituent volume fractions that lead to a continuous change in the composition, microstructure, porosity, etc. and this results in gradients in the mechanical and thermal properties [4-6].

An in-depth discussion on various issues related to FGMs on modeling aspects to determine the magnitude of thermal stresses developed due to large thermal loading, the optimal composition of the constituents or the composition profile that helps to decrease the magnitude of thermal stress, formation of cracks and eventually failure of the structural member by fracture and many other topics can be found in the article by Noda [7]. Fuchiyama and Noda [8] developed a computer program to analyze the transient heat transfer and the transient thermal stresses in FGM components by the finite element method. Obata and Noda [9] attempted studies to design an optimum functionally gradient (FG) hollow cylinder and hollow sphere in terms of reducing the thermal stresses due to uniform temperature rise and given the temperature difference between the inside and the outside. Reddy and Chin [10] developed a coupled as well as an uncoupled thermo-elastic finite element formulations to analyze the thermo-mechanical behavior of FG cylinders and plates subjected to abrupt thermal loading. Jabbari et al. [11] derived Navier's equation for an FGM hollow

cylinder taking into account the 1D steady-state heat conduction. Awaji and Sivakumar [12] numerically analyzed the steady-state and transient temperature distributions and related thermal stress distribution in an FGM cylinder composed of mullite-molybdenum system. Takezono et al. [13] carried out numerical studies on the FG cylindrical shells containing hot fluid. Ye et al. [14] discussed a 2D axisymmetric thermo-elastic problem of an FG transversely isotropic cylindrical shell. Liew et al. [15] detailed the derivation of an analytical model to carry out studies on FG hollow cylinders subjected to an arbitrary steady-state and transient temperature field.

Studies on the vibration of cylindrical shells are extensive. Many of these studies have been done on isotropic and composite shells. Among those who have carried out studies on the vibration of cylindrical shells include Arnold and Warburton [16], Ludwig and Krieg [17], Chung [18], Soedel [19], Bhimaraddi [20], Soldatos and Hajigeorgiou [21], Lam and Loy [22], and Loy, Lam and Shu [23]. In addition, several works on FGM cylindrical shells have been carried out. Najafizadeh and Isvandzibaei [24] studied the vibration of thin cylindrical shells with ring supports made of an FGM composed of stainless steel and nickel based on Third-order Shear Deformation Theory (TSDT). The influences of ring support position and boundary conditions on the frequency characteristics of thin FG cylindrical shells using various shear deformation theories have been studied by Najafizadeh and Isvandzibaei [25]. Tornabene [26] studied the dynamic behavior of moderately thick FG conical, cylindrical shells and annular plates based on the First-order Shear Deformation Theory (FSDT). Malekzadeh and Heydarpour [27] analyzed the free vibration analysis of rotating V cylindrical shells subjected to the thermal environment based on the FSDT using differential quadrature method. With the aid of generalized differential-Integral quadrature method (GDIQM), the free vibration of a two-dimensional FG circular cylindrical shell was analyzed by Ebrahimi and Najafizadeh [28]. The dynamic behavior of moderately thick FG cylindrical shell surrounded by Winkler-Pasternak elastic foundation based on the FSDT

was investigated by bahadori and najafizadeh [29]. Navier-differential quadrature solution method was used to obtain natural frequencies and mode shapes.

The literature survey revealed that most of the studies have been performed on the vibration analysis of clamped FG cylindrical shells with the numerical method. However, in this paper, an approximate analytical closed-form solution is presented for the first time to consider the vibration problem formulation for the clamped-clamped FG cylindrical shell. The FGM considered is composed of stainless steel and nickel where the volume fractions follow a power-law distribution. The objectives are to study the frequency characteristics, the influence of the constituent volume fractions, and the effects of the configurations of the constituent materials on the natural frequencies. The analysis of the FG cylindrical shell is carried out using FSDT and solved using Galerkin method.

**2. Problem formulation**

Consider a cylindrical shell with radius  $R$ , length  $L$ , and thickness  $H$ , as shown in Fig.1. The deformations, defined with reference to a coordinate system  $(x, \theta, z)$ , taken at the middle surface, are  $u, v$  and  $w$  in the  $x, \theta$  and  $z$  directions, respectively.

**2.1. The first-order shear deformation theory (FSDT)**

A cylindrical shell composed of an FGM is essentially an inhomogeneous shell consisting of a mixture of isotropic materials. It was found that the shear deformation effects of an FGM are generally more important than those for isotropic and laminated materials. Therefore, using FSDT for the vibration analysis of an FG shell is a suitable theory. In the first-order shear deformation shell theory, the Kirchhoff hypothesis is relaxed with considering the transverse normal not to be remaining perpendicular to the mid-surface after deformation. Moreover, in developing a moderately thick shell theory, the other assumptions are made, outlined as:

- The normal strain is small and negligible, so the transverse normal is considered inextensible:  $\epsilon_z=0$ .
- The shell deflections are small and the strains are infinitesimal.
- The shell is moderately thick and therefore it is possible to assume that the thickness direction normal stress is negligible so that the plane assumption can be invoked:  $\sigma_z=0$ .

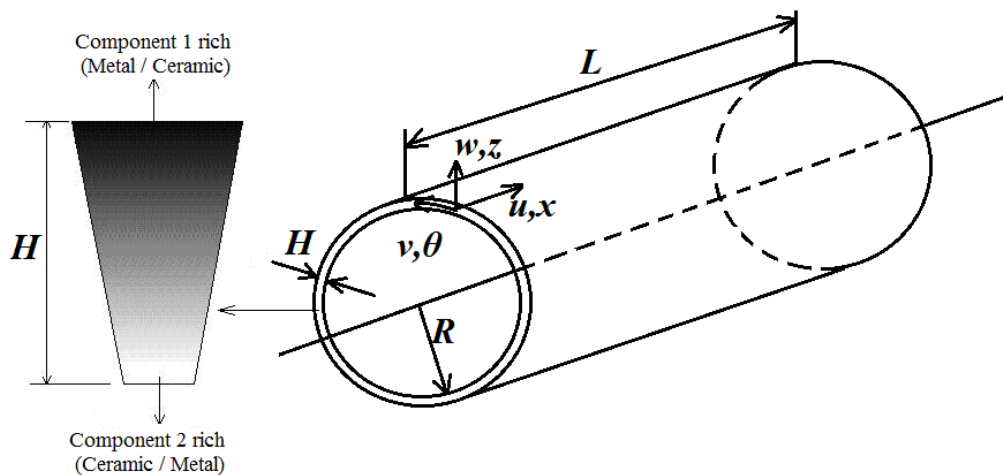


Fig. 1. An FG cylindrical shell.

- The linear elastic behavior of an FGM is assumed.
- The rotary inertia is also taken into account.

2. 2. Stress-Strain Relations

For an FG cylindrical shell that includes the effect of transverse shear deformation, the stress-strain relations can be presented in the following form:

$$\begin{Bmatrix} \sigma_x \\ \sigma_\theta \\ \tau_{x\theta} \\ \tau_{xz} \\ \tau_{\theta z} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_{x\theta} \\ \varepsilon_{xz} \\ \varepsilon_{\theta z} \end{Bmatrix} \quad (1)$$

where material constants  $Q_{ij}$  are defined as [30]:

$$\begin{aligned} Q_{11} &= \frac{E(z)}{1-\nu(z)^2}, \quad Q_{12} = \frac{\nu(z)E(z)}{A(1-\nu(z)^2)}, \\ Q_{21} &= \frac{\nu(z)E(z)}{1-\nu(z)^2}, \quad Q_{22} = \frac{E(z)}{A(1-\nu(z)^2)} \\ Q_{66} &= \frac{E(z)}{2A(1-\nu(z))}, \quad Q_{44} = K \frac{E(z)}{2(1-\nu(z))} \\ , \quad Q_{55} &= \frac{Q_{44}}{A}, \quad A = 1 + \frac{z}{R} \end{aligned} \quad (2)$$

and  $K$  is the shear correction factor that is taken as  $K=5/6$  in the present implementation of the FSDT. A typical effective material properties  $P$  of the FGM, such as  $E(z)$  or  $\nu(z)$  or density  $\rho(z)$  or thermal expansion  $\alpha(z)$  or thermal conductivity  $k(z)$  can be expressed in terms of the corresponding material properties of its individual components as:

$$P_{FGM} = P_c V_c + P_m V_m \quad (3)$$

where  $P_c$  and  $P_m$  are the material properties of the ceramic and metal, respectively. If the volume fraction of the ceramic part is represented by  $V_c$  and the metallic part by  $V_m$ , it can be followed as:

$$V_c + V_m = 1 \quad (4)$$

In this study, the properties of the shell are assumed to vary through the thickness of the shell with a power-law distribution of the volume fraction of the two materials between the two surfaces. In fact, the top surface of the shell is metal rich whereas the bottom surface is ceramic rich. Based on the power-law distribution [27], the variation of  $V_c$  versus thickness coordinate,  $z$ , placed at the reference surface can be expressed as:

$$V_c(z) = \left( \frac{2z + h}{2h} \right)^g, \quad g \geq 0 \quad (5)$$

where  $g$  is the power-law index. Under the assumptions of a moderately thick shell theory reported above, the displacement field of the first-order theory is of the form:

$$\begin{aligned} u(x, \theta, z, t) &= u_0(x, \theta, t) + z \psi_x(x, \theta, t), \\ v(x, \theta, z, t) &= v_0(x, \theta, t) + z \psi_\theta(x, \theta, t), \\ w(x, \theta, z, t) &= w_0(x, \theta, t) \end{aligned} \quad (6)$$

where  $(u_0, v_0, w_0, \psi_x, \psi_\theta)$  are unknown functions to be determined. The kinematics hypothesis is expressed by Eq. (3) should be supplemented by the statement that the shell deflections are small and strains are infinitesimal, that is  $w(x, \theta, t) \ll h$ .

A relationship between strains and displacements, in terms of the cylindrical coordinates, can be expressed as [31]:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{x\theta} \\ \varepsilon_{xz} \\ \varepsilon_{\theta z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{1}{R} \frac{\partial}{\partial \theta} & \frac{1}{R} \\ \frac{1}{R} \frac{\partial}{\partial \theta} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{1}{R} \frac{\partial}{\partial \theta} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} \quad (7)$$

Considering the first order shear deformation theory and substituting Eq. (6) into Eq. (7), it can be deduced:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{x\theta} \\ \varepsilon_{xz} \\ \varepsilon_{\theta z} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{\theta\theta}^0 \\ \varepsilon_{x\theta}^0 \\ \varepsilon_{xz}^0 \\ \varepsilon_{\theta z}^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{\theta\theta}^1 \\ \varepsilon_{x\theta}^1 \\ \varepsilon_{xz}^1 \\ \varepsilon_{\theta z}^1 \end{Bmatrix} \quad (8)$$

where

$$\begin{Bmatrix} \varepsilon^0 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{\theta\theta}^0 \\ \varepsilon_{x\theta}^0 \\ \varepsilon_{xz}^0 \\ \varepsilon_{\theta z}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{w_0}{R} + \frac{\partial v_0}{R \partial \theta} \\ \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R \partial \theta} \\ \psi_x + \frac{\partial w_0}{\partial x} \\ \psi_\theta + \frac{\partial w_0}{R \partial \theta} \end{Bmatrix}, \quad (9)$$

$$\begin{Bmatrix} \varepsilon^1 \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{\theta\theta}^1 \\ \varepsilon_{x\theta}^1 \\ \varepsilon_{xz}^1 \\ \varepsilon_{\theta z}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \psi_x}{\partial x} \\ \frac{\partial \psi_\theta}{R \partial \theta} \\ 0 \\ 0 \\ \frac{\partial \psi_\theta}{\partial x} + \frac{\partial \psi_x}{R \partial \theta} \end{Bmatrix}$$

### 2. 3. Equations of motion

The governing differential equations of motion can be derived, using Hamilton's principle as follow:

$$\int_{t_1}^{t_2} (\delta T_{sh} - \delta U_{sh}^\varepsilon + \delta W) dt = 0 \quad (10)$$

where  $\delta T_{sh}$  and  $\delta U_{sh}^\varepsilon$  are the variations of the kinetic energy and strain energy of the shell, respectively.  $\delta W$  is the variations of the thermal work and  $t$  denotes the time.

The strain energy of the FG cylindrical shell is expressed as:

$$U_{sh}^\varepsilon = \frac{1}{2} \int_{-\frac{H}{2}}^{\frac{H}{2}} \int_0^L \int_0^{2\pi} \{\varepsilon\}^T [Q] \{\varepsilon\} (R+z) d\theta dx dz \quad (11)$$

The kinetic energy of the rotating cylindrical shell is expressed as:

$$T_{sh} = \frac{1}{2} \int_{-\frac{H}{2}}^{\frac{H}{2}} \int_0^L \int_0^{2\pi} \rho(z) \left\{ \begin{aligned} &(\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + z^2 \dot{\psi}_x^2 + z^2 \dot{\psi}_\theta^2) \\ &+ 2z\dot{u} \dot{\psi}_x + 2z\dot{v} \dot{\psi}_\theta \end{aligned} \right\} (R+z) dx d\theta dz \quad (12)$$

For simplicity, by solving the membrane form of equilibrium equations, it can be found that  $N_x^T \neq 0$ ,  $N_\theta^T = N_{x\theta}^T = M_x^T = M_\theta^T = M_{x\theta}^T = 0$  [32]. The work carried out on the shell due to thermal load is described as:

$$W = \frac{N_x^T}{2} \int_0^L \int_0^{2\pi} \left[ \left( \frac{\partial w}{\partial x} \right)^2 \right] (R+z) dx d\theta \quad (13)$$

where  $N_x^T$  is defined as:

$$N_x^T = \int_{-\frac{H}{2}}^{\frac{H}{2}} \beta \Delta T(z) dz, \quad \beta = [Q_{11}(z) + Q_{12}(z)] \alpha(z)$$

$$, \quad \Delta T(z) = T(z) - T_m \quad (14)$$

where  $\Delta T(z)$  is the temperature change and  $T_m$  is the outer surface temperature.

The temperature variation is assumed to be varied in thickness direction only; therefore, it can be described in terms of the solution of the steady-state heat transfer equation of the FG cylindrical shell. The equation of the temperature field is given by:

$$-\frac{d}{dz} \left[ k(z) \frac{dT}{dz} \right] = 0 \quad (15)$$

This equation is solved with imposing boundary condition of  $T=T_c$  at  $z=-h/2$  and  $T=T_m$  at  $z=h/2$ .

The solution of this equation, by means of polynomial series, is [33]:

$$T(z) = T_c + (T_m - T_c)\eta(z) \tag{16}$$

and

$$\eta(z) = \frac{1}{C} \left[ \begin{aligned} &\left(\frac{2z+h}{2h}\right) - \frac{(k_m - k_c)}{(g+1)k_c} \left(\frac{2z+h}{2h}\right)^{g+1} \\ &+ \frac{(k_m - k_c)^2}{(2g+1)k_c^2} \left(\frac{2z+h}{2h}\right)^{2g+1} \\ &- \frac{(k_m - k_c)^3}{(3g+1)k_c^3} \left(\frac{2z+h}{2h}\right)^{3g+1} \\ &+ \frac{(k_m - k_c)^4}{(4g+1)k_c^4} \left(\frac{2z+h}{2h}\right)^{4g+1} \\ &- \frac{(k_m - k_c)^5}{(5g+1)k_c^5} \left(\frac{2z+h}{2h}\right)^{5g+1} \end{aligned} \right] \tag{17}$$

$$C = 1 - \frac{(k_m - k_c)}{(g+1)k_c} + \frac{(k_m - k_c)^2}{(2g+1)k_c^2} - \frac{(k_m - k_c)^3}{(3g+1)k_c^3} + \frac{(k_m - k_c)^4}{(4g+1)k_c^4} - \frac{(k_m - k_c)^5}{(5g+1)k_c^5} \tag{18}$$

Substituting Eqs. (11-13) into Eq. (10) and then applying Hamilton's principle to the energy functional yields the motion equations as [34]:

$$\begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{x\theta}}{R\partial\theta} &= I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \psi_x}{\partial t^2} \\ \frac{\partial N_{x\theta}}{\partial x} + \frac{\partial N_\theta}{R\partial\theta} + \frac{1}{R} Q_\theta &= I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \psi_\theta}{\partial t^2} \\ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_\theta}{R\partial\theta} - \frac{N_\theta}{R} + N^T \frac{\partial^2 w_0}{\partial x^2} &= I_0 \frac{\partial^2 w_0}{\partial t^2} \\ \frac{\partial M_x}{\partial x} + \frac{\partial M_{x\theta}}{R\partial\theta} - Q_x &= I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^2 \psi_x}{\partial t^2} \\ \frac{\partial M_{x\theta}}{\partial x} + \frac{\partial M_\theta}{R\partial\theta} - Q_\theta &= I_1 \frac{\partial^2 v_0}{\partial t^2} + I_2 \frac{\partial^2 \psi_\theta}{\partial t^2} \end{aligned} \tag{19}$$

where

$$\begin{aligned} (I_0, I_1, I_2) &= \int_{-H/2}^{H/2} (1, z, z^2) \rho dZ \\ N_x &= A_{11} \frac{\partial u_0}{\partial x} + A_{12} \left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right) + \frac{B_{11}}{R} \frac{\partial \psi_x}{\partial x} + \frac{B_{12}}{R} \frac{\partial \psi_\theta}{R\partial\theta} \\ N_\theta &= A_{12} \frac{\partial u_0}{\partial x} + A_{22} \left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right) + \frac{B_{12}}{R} \frac{\partial \psi_x}{\partial x} + \frac{B_{22}}{R} \frac{\partial \psi_\theta}{R\partial\theta} \\ N_{x\theta} &= A_{66} \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R\partial\theta}\right) + \frac{B_{66}}{R} \left(\frac{\partial \psi_\theta}{\partial x} + \frac{\partial \psi_x}{R\partial\theta}\right) \\ M_x &= B_{11} \frac{\partial u_0}{\partial x} + B_{12} \left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right) + D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_\theta}{R\partial\theta} \\ M_\theta &= B_{12} \frac{\partial u_0}{\partial x} + B_{22} \left(\frac{w_0}{R} + \frac{\partial v_0}{R\partial\theta}\right) + D_{12} \frac{\partial \psi_x}{\partial x} + D_{22} \frac{\partial \psi_\theta}{R\partial\theta} \\ M_{x\theta} &= B_{66} \left(\frac{\partial v_0}{\partial x} + \frac{\partial u_0}{R\partial\theta}\right) + D_{66} \left(\frac{\partial \psi_\theta}{\partial x} + \frac{\partial \psi_x}{R\partial\theta}\right) \\ Q_x &= K A_{44} \left(\psi_x + \frac{\partial w_0}{\partial x}\right) \\ Q_\theta &= K A_{55} \left(\psi_\theta + \frac{\partial w_0}{R\partial\theta}\right) \end{aligned} \tag{20}$$

where  $\{A_{ij}, B_{ij}, D_{ij}\} = \int_{-H/2}^{H/2} Q_{ij} \{1, z, z^2\} dz$ .

### 2. 4. Boundary condition

In the present paper, one of the following boundary conditions is considered as follows:

- a) Simply supported:
- b)

$$v = w = N_x = M_x = N_{x\theta} = 0 \tag{21}$$

The displacement fields which satisfy these boundary conditions can be written as:

$$\begin{aligned} u &= \tilde{A} e^{i\alpha x} \text{Cos}\left(\frac{m\pi x}{L}\right) \text{Cos}(n\theta), \\ v &= \tilde{B} e^{i\alpha x} \text{Sin}\left(\frac{m\pi x}{L}\right) \text{Sin}(n\theta), \\ w &= \tilde{C} e^{i\alpha x} \text{Sin}\left(\frac{m\pi x}{L}\right) \text{Cos}(n\theta) \\ \psi_x &= \tilde{D} e^{i\alpha x} \text{Cos}\left(\frac{m\pi x}{L}\right) \text{Cos}(n\theta), \\ \psi_\theta &= \tilde{E} e^{i\alpha x} \text{Sin}\left(\frac{m\pi x}{L}\right) \text{Sin}(n\theta) \end{aligned} \tag{22}$$

- b) Clamped:

$$u = v = w = \psi_x = \psi_\theta = 0 \tag{23}$$

The displacement field which satisfy these boundary conditions can be written as [35]:

$$\begin{aligned}
 u &= \tilde{A}H(x)\text{Cos}(n\theta)e^{i\omega t}, v = \tilde{B}\phi(x)\text{Sin}(n\theta)e^{i\omega t}, \\
 w &= \tilde{C}\phi(x)\text{Cos}(n\theta)e^{i\omega t} \\
 \psi_x &= \tilde{D}H(x)\text{Cos}(n\theta)e^{i\omega t}, \\
 \psi_\theta &= \tilde{E}\phi(x)\text{Sin}(n\theta)e^{i\omega t} \\
 H(x) &= \frac{x}{L}(2\frac{x^2}{L^2} - 3\frac{x}{L} + 1), \\
 \phi(x) &= \frac{x}{L}(\frac{x}{L} - 1)[\frac{x}{L}(\frac{x}{L} - 1)]
 \end{aligned}
 \tag{24}$$

2. 5. Solution method

When Eqs .(22, 24) are substituted into the differential Eq. (19), the residuals  $F_1, F_2, F_3, F_4,$  and  $F_5$  are derived. According to the Galerkin method, the residuals are multiplied by the comparison functions, integrate the product over the domain of the system, and equate the result to zero:

$$\begin{aligned}
 \int_0^{2\pi} \int_0^L F_1 u_0 dx R d\theta &= 0 \\
 \int_0^{2\pi} \int_0^L F_2 v_0 dx R d\theta &= 0 \\
 \int_0^{2\pi} \int_0^L F_3 w_0 dx R d\theta &= 0 \\
 \int_0^{2\pi} \int_0^L F_3 \psi_x dx R d\theta &= 0 \\
 \int_0^{2\pi} \int_0^L F_3 \psi_\theta dx R d\theta &= 0
 \end{aligned}
 \tag{25}$$

Upon integration, Eq. (25) denote a set of linear homogeneous algebraic equations in the unknown coefficients  $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$  and  $\tilde{E}$  in the following form:

$$\begin{aligned}
 a_{11}\tilde{A} + a_{12}\tilde{B} + a_{13}\tilde{C} + a_{14}\tilde{D} + a_{15}\tilde{E} &= 0 \\
 a_{21}\tilde{A} + a_{22}\tilde{B} + a_{23}\tilde{C} + a_{24}\tilde{D} + a_{25}\tilde{E} &= 0 \\
 a_{31}\tilde{A} + a_{32}\tilde{B} + a_{33}\tilde{C} + a_{34}\tilde{D} + a_{35}\tilde{E} &= 0 \\
 a_{41}\tilde{A} + a_{42}\tilde{B} + a_{43}\tilde{C} + a_{44}\tilde{D} + a_{45}\tilde{E} &= 0 \\
 a_{51}\tilde{A} + a_{52}\tilde{B} + a_{53}\tilde{C} + a_{54}\tilde{D} + a_{55}\tilde{E} &= 0
 \end{aligned}
 \tag{26}$$

For suitability in determining, rearranging Eq. (26) as stated in  $\omega$ , the eigenvalue problem is achieved on the subsequent matrix form,

$$[M\omega^2 + K]d = 0
 \tag{27}$$

here  $d = [\tilde{A} \ \tilde{B} \ \tilde{C} \ \tilde{D} \ \tilde{E}]^T$ .

Eq. (27) can be solved using a traditional eigenvalue approach, and six  $\omega_i$  are attained.

3. Results and discussion

Here, the results of natural frequencies of an FG cylindrical shell made up of stainless steel and nickel are presented. Also, two boundary conditions as noted previously are considered here for the cylindrical shell. The material properties used in the present study are given in Table 1.

Tables 2 and 3 demonstrate a comparison between frequencies of FG cylindrical shells made up of stainless steel –zirconia using ABAQUS model and the present method. As can be seen, there is a good conformity between the results with a maximum less than 4% deviations. In this study, the difference is defined as:

$$\text{Difference}(\%) = \frac{Abaqus - Galerkin}{Abaqus} \times 100$$

Table 1. Material properties.

Material	E (Pa)	$\nu$	$\rho$ (kg/m <sup>3</sup> )	K (W/mK <sup>o</sup> )	$\alpha$ (1/C <sup>o</sup> )
Stainless Steel	207.778e9	0.317756	8166	15.379	12.33e-6
Nickel	205.098e9	0.31	8900	187.66	9.9209e-6
Steel	201.04e9	0.3262	8166	15.379	12.33e-6
Zirconium	244.27e9	0.2882	5700	1.7	12.766e-6
Aluminium	70 e9	0.3	2707	204	23e-6
Zirconia	151e9	0.3	3000	2.09	10e-6

**Table 2.** Comparison of frequencies (HZ) of FG cylindrical shells with simply supported boundary conditions steel-zirconia ( $L/R=20, H/R=0.002, g=0.1$ ).

No	ABAQUS	Galerkin	Mode No.	Difference (%)
1	4.3461	4.3118	(1,3)	0.79
2	4.6265	4.6297	(1,2)	-0.07
3	7.4467	7.3080	(1,4)	1.81
4	8.8257	8.7	(2,4)	1.42
5	9.1710	9.1475	(2,3)	0.26

**Table 3.** Comparison of frequencies (HZ) of FG cylindrical shells with Simply Supported boundary conditions steel-zirconia ( $L/R=3, H/R=0.002, g=0.1$ ).

No	ABAQUS	Galerkin	Mode No.	Difference (%)
1	30.199	29.9582	(1,6)	0.8
2	31.194	30.1875	(1,7)	3.23
3	35.918	34.6003	(1,8)	3.67
4	36.825	36.0782	(1,5)	2.03
5	45.594	43.5913	(1,9)	2.2
6	51.17	50.2969	(1,4)	1.71

**Table 4.** Comparison of frequencies (HZ) of FG cylindrical shells with Clamped boundary conditions steel-zirconia ( $L/R=3, H/R=0.002, g=0.1$ ).

No	ABAQUS	Galerkin	Mode No.	Difference (%)
1	45.094	45.3476	(1,7)	-0.56
2	45.121	43.8775	(1,8)	2.83
3	50.353	47.0195	(1,9)	7.09
4	51.509	52.7234	(1,6)	-2.3
5	59.562	53.4622	(1,10)	11.42
6	65.674	67.4569	(1,11)	-2.64

Table 4 demonstrates a comparison between frequencies of FG cylindrical shells made up of Stainless Steel–Zirconia using ABAQUS model and present method.

Figure 0 demonstrates a comparison between results of recent article and frequencies of FG cylindrical shells made up of aluminium–zirconia subjected to temperature gradient with DQM [36] and exact solution [30]. As can be seen, there is a total conformity between these results.

Figure 3 illustrates the effects of the circumferential wave number on the frequencies of the clamped FG cylindrical shell with different thicknesses of the shell, namely,  $H = 2$  mm and 20 mm. From the figures, it can be observed that, at small circumferential wave number, i.e.,  $n$  equals to 1 and 2, the influence of thickness is insignificant. At large circumferential wave number  $n$ , the frequency increases with rising the thickness of the shell. However, the number of circumferential waves with the occurrence of the fundamental

frequency decreases when the thickness is enhanced. For example, the fundamental frequency occurs at  $n=6$  for  $H=0.002$ , and at  $n=3$  for  $H=0.02$  for a shell with  $L=2$  m.

Figure 4 shows the effect of thermal loads on the fundamental frequency of a clamped FG cylindrical shell in both different length shells. It is seen from these figures that the fundamental frequency decreases as the temperature difference between the inner and outer surface increases. It is also observed that the fundamental frequency mode does not shift when the  $T_{io}$  is increased. In addition, it is noteworthy that, the level of the sensitivity of fundamental frequency to  $T_{io}$  increases by raising the length of the shell. For example, the discrepancies of results are 21.7% and 24.3% for shells with  $L=2$ m and  $L=3$ m, respectively. In this comparison, the discrepancy is defined as:

$$Discrepancy (\%) = \frac{freq_{T_{io}=100} - freq_{T_{io}=250}}{freq_{T_{io}=100}} \times 100$$



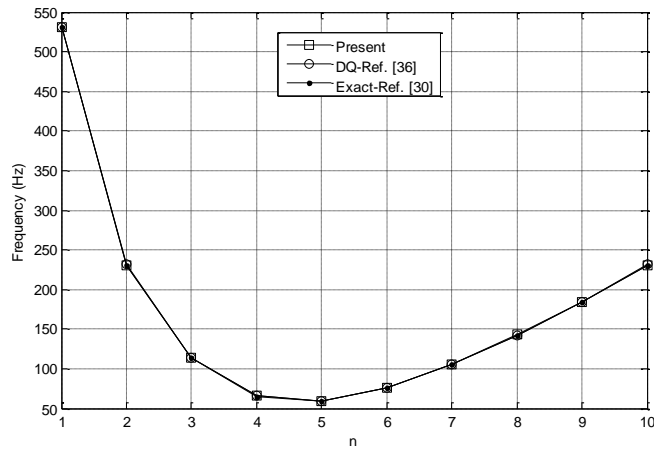


Fig. 2. Comparison of frequency for an FG cylindrical shell aluminium-zirconia ( $g=1, H=0.002, L=2, R=0.5$ ).

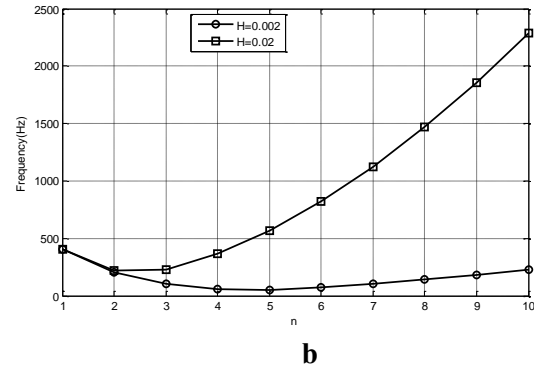
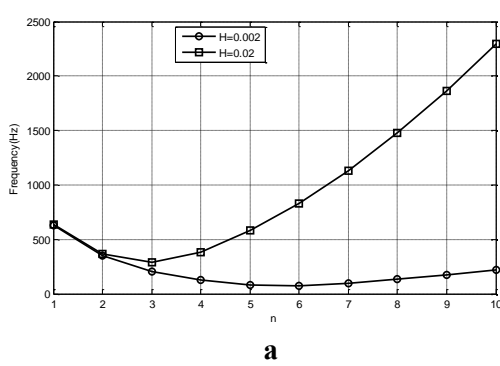


Fig. 3. Effect of thickness on natural frequencies of a clamped FG cylindrical shell. a)  $L=2m$ , b)  $L=3m$  ( $T_{i0}=250(C^\circ)$ ).

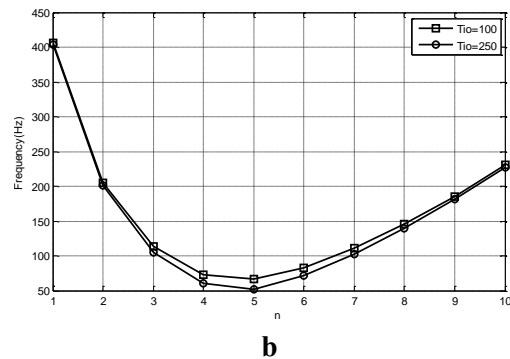
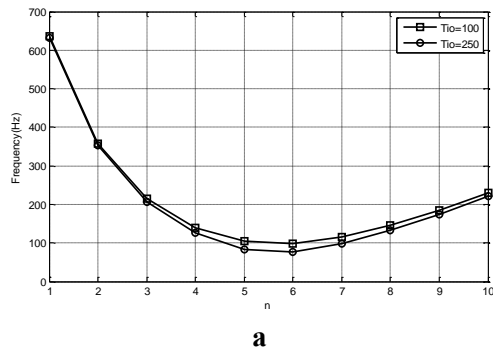


Fig. 4. Effect of temperature gradient on natural frequencies of a clamped FG cylindrical shell. a)  $L=2m$ , b)  $L=3m$  ( $H=0.002$ ).

4. Conclusions

There is no exact solution for a cylindrical shell with fully clamped boundary condition. Therefore, the governing equations are solved

using Galerkin Method. From the study, some conclusions can be made particularly as:

1. As could be predicted, with increasing  $T_{i0}$  natural frequencies of all modes are decreased.

2. The number of circumferential waves with the occurrence of the fundamental frequency decreases when the thickness is enhanced.
3. The fundamental frequency mode does not change when the  $T_{i0}$  is enhanced.
4. The level of the sensitivity of fundamental frequency to  $T_{i0}$  increases by raising the length of the shell.
5. The effect of  $T_{i0}$  on natural frequency of an FG shell is significant when the circumferential wave number is near to the fundamental frequency mode.

## References

- [1] M. Yamanouchi, M. Koizumi, T. Hirai, and I. Shiota. *Proceedings of the First International Symposium on Functionally Gradient Materials*, Japan, (1990).
- [2] M. Koizumi, "The concept of FGM", *Ceramic Transactions, Functionally Gradient Materials*, Vol. 34, pp. 3-10, (1993).
- [3] Anon, FGM components: PM meets the challenge. *Metal Powder Report*; Vol. 51, pp. 28-32, (1996).
- [4] N. Sata, "Characteristic of SiC-TiB<sub>2</sub> composites as the surface layer of SiC-TiB<sub>2</sub>-Cu functionally gradient material produced by self-propagating high-temperature synthesis", *Ceramic Transactions, Functionally Gradient Materials*, Vol. 34, pp. 109-116, (1993).
- [5] H. Yamaoka, M. Yuki, K. Tahara, T. Irisawa, R. Watanabe, and A. Kawasaki. "Fabrication of Functionally Gradient Material by slurry stacking and sintering process", *Ceramic Transactions, Functionally Gradient Materials*, Vol. 34, pp. 72-165, (1993).
- [6] B. H. Rabin, and R. J. Heaps, "Powder processing of Ni/Al<sub>2</sub>O<sub>3</sub> FGM", *Ceramic Transactions, Functionally Gradient Materials*, Vol. 34, pp. 173-180, (1993).
- [7] N. Noda, "Thermal stresses in functionally graded materials", *Journal of Thermal Stresses* Vol. 22, pp. 477-512, (1999).
- [8] T. Fuchiyama, and N. Noda, "Analysis of thermal stress in a plate of functionally gradient material", *Journal of Science and Engineering*, Vol. 16, pp. 263-268, (1995).
- [9] Y. Obata, and N. Noda, "Steady thermal stresses in a hollow circular cylinder and hollow sphere of a functionally gradient material", *Journal of Thermal Stresses*, Vol. 17, pp. 471-487, (1994).
- [10] J. N. Reddy, and C. D. Chin, "Thermo mechanical analysis of functionally graded cylinders and plates", *Journal of Thermal Stresses*, Vol. 21, pp. 593-626, (1998).
- [11] M. Jabbari, S. Sohrabpour, and M. R. Eslami, "Mechanical and thermal stresses in a functionally graded hollow cylinder due to radially symmetric load", *International Journal of Pressure Vessels and Piping*, Vol. 79, pp. 493-497, (2002).
- [12] H. Awaji, and R. Sivakumar, "Temperature and stress distribution in a hollow cylinder of functionally graded material; the case of temperature-independent material properties", *Journal of the American Ceramic Society*, Vol. 84, pp. 1059-1065, (2001).
- [13] S. Takezono, K. Tao, E. Inamura, and M. Inoue, "Thermal stress and deformation in functionally graded material shells of revolution under thermal loading due to fluid", *JSME International Journal Series*, Vol. 39, pp. 573-581, (1996).
- [14] G. R. Ye, W. Q. Chen, and J. B. Cai, "A uniformly heated functionally graded cylindrical shell with transverse isotropy", *Mechanics Research Communications*, Vol. 28, pp. 535-542, (2001).
- [15] K. M. Liew, S. Kitipornchai, X. Z. Zhang, and C. W. Lim, "Analysis of the thermal stress behavior of functionally graded hollow circular cylinders", *International Journal of Solids and Structures*, Vol. 40, pp. 2355-2380, (2003).
- [16] R. N. Arnold, and G. B. Warburton, "Flexural vibrations of the walls of thin cylindrical shells having freely supported ends", *Proceedings of the Royal Society London A*, Vol. 197, pp. 238-256, (1949).

- [17] A. Ludwig, and R. Krieg, "An analytical quasi-exact method for calculating Eigen vibrations of thin circular cylindrical shells", *Journal of Sound and Vibration*, Vol. 74, pp. 155-174, (1981).
- [18] H. Chung, "Free vibration analysis of circular cylindrical shells", *Journal of Sound and Vibration*, Vol. 74, pp. 331-350, (1981).
- [19] W. Soedel, "A new frequency formula for closed circular cylindrical shells for a large variety of boundary conditions", *Journal of Sound and Vibration*, Vol. 70, pp. 309-317, (1980).
- [20] A. Bhimaraddi, "A higher order theory for free vibration analysis of circular cylindrical shells", *International Journal of Solids and Structures*, Vol. 20, pp. 623-630, (1984).
- [21] K. P. Soldatos, and V. P. Hajigeorou, "Three-dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels", *Journal of Sound and Vibration*, Vol. 137, pp. 369-384, (1990).
- [22] K. Y. Lam, and C. T. Loy, "Effects of boundary conditions on frequencies characteristics for a multi-layered cylindrical shell", *Journal of Sound and Vibration*, Vol. 188, pp. 363-384, (1995).
- [23] C. T. Loy, K. Y. Lam, and C. Shu, "Analysis of cylindrical shells using generalized differential quadrature", *Shock and Vibration*, Vol. 4, pp. 193-198, (1997).
- [24] MM. Najafizadeh, MR. Isvandzibaei, "Vibration of functionally graded cylindrical shells based on higher order shear deformation plate theory with ring support. *Acta Mechanica* 2007; 191: 75-91.
- [25] M. M. Najafizadeh, and M. R. Isvandzibaei, "Vibration of functionally graded cylindrical shells based on different shear deformation shell theories with ring support under various boundary conditions", *Journal of Mechanical Science and Technology*, Vol. 23, pp. 2072-2084, (2009).
- [26] F. Tornabene, "Free vibration analysis of functionally graded conical, cylindrical shell and annular plate structures with a four-parameter power-law distribution", *Comput. Methods Appl. Mech. Engrg.*, Vol. 198, pp. 2911-2935, (2009).
- [27] P. Malekzadeh, and Y. Heydarpour, "Free vibration analysis of rotating functionally graded cylindrical shells in thermal environment", *Composite Structures*, Vol. 94, pp. 2971-2981, (2012).
- [28] M. J. Ebrahimi, and M. M. Najafizadeh, "Free vibration of two-dimensional functionally graded circular cylindrical shells on elastic foundation", *Modares Mechanical Engineering*, Vol. 38, No. 1, pp. 308-324, (2013).
- [29] R. Bahadori, and M. M. Najafizadeh "Free vibration analysis of two-dimensional functionally graded axisymmetric cylindrical shell on Winkler–Pasternak elastic foundation by First-order Shear Deformation Theory and using Navier-differential quadrature solution methods", *Applied Mathematical Modelling*, Vol. 39, pp. 4877-4894, (2015).
- [30] G. G. Sheng and X. Wang, "Effects of Thermal Loading on the Buckling and Vibration of Ring-Stiffened Functionally Graded Shell", *J. Therm. Stresses*, Vol. 30, pp. 1249-1267, (2007).
- [31] K. Y. Lam, and W. Qian, "Vibrations of Thick Rotating Laminated Composite Cylindrical Shells", *J. Sound Vibr.*, Vol. 225, No. 3, pp. 483-501, (1999).
- [32] R. Naj, M. Sabzikar Boroujerdy and M. R. Eslami, "Thermal and mechanical instability of functionally graded truncated conical shells", *Thin-Walled Structures*, Vol. 46, pp. 65-78, (2008).
- [33] H.-S. Shen, and N. Noda, "Postbuckling of FGM Cylindrical Shells under Combined Axial and Radial Mechanical Loads in Thermal Environments", *Int. J. Solids Struct.*, Vol. 42, pp. 4641-4662, (2005).

- [34] M. S. Qatu, “*Vibration of Laminated Shells and Plates*”, Elsevier, The Netherlands, (2004).
- [35] A. V. Lopatin, and E. V. Morozov, "Buckling of the composite sandwich cylindrical shell with clamped ends under uniform external pressure", *Compos. Struct.*, Vol. 122, pp. 209-216, (2015).
- [36] M. Talebitooti, “Vibration and critical speed of orthogonally stiffened rotating FG cylindrical shell under thermo-mechanical loads using differential quadrature method” *J. Term. Stresses*, Vol. 36, pp.160-188, (2013).

**How to cite this paper:**

M. Talebitooti, M. Ghasemi, S. M. Hosseini, “Vibration analysis of functionally graded cylindrical shells with different boundary conditions subjected to thermal loads”, *Journal of Computational and Applied Research in Mechanical Engineering*, Vol. 6. No. 2, pp. 103-114

**DOI:** 10.22061/jcarme.2017.602

**URL:** [http://jcarme.srttu.edu/?\\_action=showPDF&article=602](http://jcarme.srttu.edu/?_action=showPDF&article=602)

