



On borderenergetic and L-borderenergetic graphs

Mardjan Hakimi-Nezhaad *

Department of Mathematics, Faculty of Science, Shahid Rajaei Teacher Training University, Tehran, 16785 - 136, I. R. Iran

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Abstract. A graph G of order n is said to be borderenergetic if its energy is equal to $2n - 2$. In this paper, we study the borderenergetic and Laplacian borderenergetic graphs.

Keywords. energy (of graph), adjacency matrix, Laplacian matrix, signless Laplacian matrix.

1 Introduction

We first recall some definitions that will be kept throughout. Let G be a simple graph with n vertices and $m(G)$ edges, and $A(G)$ denotes its adjacency matrix. Let $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ be the Laplacian and signless Laplacian matrix of the graph G , respectively, where $D(G) = [d_{ij}]$ is the diagonal matrix whose entries are the degree of vertices, i.e., $d_{ii} = \text{deg}(v_i)$ and $d_{ij} = 0$ for $i \neq j$.

The energy of G is a graph invariant which was introduced by Ivan Gutman [6]. It is defined as $E(G) = \sum_{i=1}^n |\lambda_i|$, where λ_i 's are eigenvalues of G . If $0 = \mu_1 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \mu_n$ and $q_1 \leq q_2 \leq \dots \leq q_{n-1} \leq q_n$ are the Laplacian and signless Laplacian eigenvalues of G then the quantities $E_L(G) = \sum_{i=1}^n |\mu_i - \frac{2m(G)}{n}|$ and $E_Q(G) = \sum_{i=1}^n |q_i - \frac{2m(G)}{n}|$ are called the Laplacian and signless Laplacian energy of G , respectively. Details on the properties of Laplacian and signless Laplacian energy can be found in [6, 8, 13].

The first borderenergetic graph was discovered by Hou et al. in 2001 [11], but in that time it did not attract much attention. Recently, Gong et al in [5] studied the graphs with the same

*Corresponding author (Email address: m.hakiminezhaad@sru.ac.ir).

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energy as a complete graph. They put forward the concept of borderenergetic graphs.

A graph G on n vertices is said to be borderenergetic if its energy equals the energy of the complete graph K_n , i.e., if $E(G) = E(K_n) = 2(n - 1)$. In [5], it was shown that there exist borderenergetic graphs on order n for each integer $n \geq 7$. The number of borderenergetic graphs were determined for $n = 7, 8, 9$ [5], $n = 10, 11$ [14, 17] and $n = 12$ [4].

In [12], a family of non-regular and non-integral borderenergetic threshold graphs was discovered. In [3], the authors obtained three asymptotically tight bounds on the number of edges of borderenergetic graphs. We refer the readers to [9, 15] for more information.

An analogous concept as borderenergetic graphs, called Laplacian borderenergetic graphs was proposed in [19]. That is, a graph G of order n is Laplacian borderenergetic or L -borderenergetic for short, if $E_L(G) = E_L(K_n) = 2n - 2$.

In [1], Deng et al. presented some asymptotically bounds on the order and size of L -borderenergetic graphs. Also, they showed that all trees, cycles, the complete bipartite graphs, and many 2-connected graphs are not L -borderenergetic. They showed in [2], a kind of threshold graphs are L -borderenergetic.

Lu et al. in [16] presented all non-complete L -borderenergetic graphs of order $4 \leq n \leq 7$ and they constructed one connected non-complete L -borderenergetic graph on n vertices for each integer $n \geq 4$, which extends the result in [20] and completely confirms the existence of non-complete L -borderenergetic graphs. Particularly, they proved that there are at least $\frac{n}{2} + 4$ non-complete L -borderenergetic graphs of order n for any even integer $n \geq 6$.

Hakimi-Nezhaad et al. in [10] generalized the concept of borderenergetic graphs for the signless Laplacian matrices of graphs. That is, a graph G of order n is signless Laplacian borderenergetic or Q -borderenergetic for short, if $E_Q(G) = E_Q(K_n) = 2n - 2$. Also, they constructed sequences of Laplacian borderenergetic non-complete graphs by means of graph operations, and all the non-complete and pairwise non-isomorphic L -borderenergetic and Q -borderenergetic graphs of small order n are depicted for n with $4 \leq n \leq 9$, see Appendix.

Tao et al. in [18] considered the extremal number of edges of non-complete L -borderenergetic graph, then use a computer search to find out all the L -borderenergetic graphs on no more than 10 vertices.

Main Results

Here, we present some basic theorem used to study borderenergetic and L -borderenergetic and Q -borderenergetic graphs.

Theorem 1.1. *We have the following statements:*

- 1) [5]. *There are no borderenergetic graphs of order $n \leq 6$.*

- 2) [5]. There exists a unique borderenergetic graph of order 7.
- 3) [5]. For any $n \geq 7$, there exist borderenergetic graphs of order n .
- 4) [5]. There are exactly 6 borderenergetic graphs of order 8.
- 5) [5]. There are exactly 17 borderenergetic graphs of order 9.
- 6) [14,17]. There are exactly 49 borderenergetic graphs of order 10.
- 7) [17]. There are exactly 158 borderenergetic graphs of order 11, of which 157 are connected.
- 8) [4]. There are exactly 572 connected borderenergetic graphs of order 12.
- 8) [5]. For each integer n ($n \geq 13$), there exists a non-complete borderenergetic graph of order n .

Theorem 1.2. [9]. A borderenergetic graph of order n must possess at least $2n - 2$ edges.

Theorem 1.3. [3]. Let G be a k -regular integral graph of order n with t non-negative eigenvalues. If $E(G) = 2(n - t + k)$ then $E(\bar{G}) = 2(n - 1)$, where \bar{G} is complement of graph G .

Theorem 1.4. [15].

- 1) There is no noncomplete borderenergetic graph with maximum degree $\Delta = 2$ or 3.
- 2) Let G be a non-complete borderenergetic graph of order n with maximum degree $\Delta = 4$. Then G must have the following properties:
 - (i) $e(G) = 2n$ or $2n - 1$;
 - (ii) $|G| \leq 21$;
 - (iii) G is non-bipartite;
 - (iv) the nullity, i.e., the multiplicity of eigenvalue 0, of G is 0.
- 3) Let G be a 4-regular non-complete borderenergetic graph of order n and H is a maximal bipartite subgraph of G . Then $m(G) - m(H) \geq 3$.

Theorem 1.5. [15]. No borderenergetic graphs have minimum degree $n - 2$. Besides, for each integer $n \geq 7$, there exists a connected noncomplete borderenergetic graph of order n with minimum degree $n - 3$ and for each even integer $n \geq 8$, there exists a noncomplete borderenergetic graph of order n with minimum degree $n - 4$.

Theorem 1.6. We have the following statements:

- 1) [10]. There are exactly two non-complete L-borderenergetic disconnected graphs of orders 4 and 5, respectively.
- 2) [10]. There are exactly five non-complete L-borderenergetic disconnected graphs of order 6.

- 3) [10]. There are exactly five non-complete L -borderenergetic disconnected graphs of order 7.
- 4) [18]. There are totally 18 L -borderenergetic connected graphs on less than 8 vertices.
- 5) [10,18]. There are exactly 31 L -borderenergetic connected graphs and 27 disconnected graphs of order 8.
- 6) [10,18]. There are exactly 16 L -borderenergetic graphs and 26 disconnected graphs of order 9.
- 7) [10,18]. There are exactly 120 L -borderenergetic connected graphs on 10 vertices.

Theorem 1.7. [10] There is no Laplacian borderenergetic tree with $n \geq 3$ vertices.

Theorem 1.8. [1]. If G is a complete bipartite graph $K_{a,b}(1ab)$, then G is not L -borderenergetic.

Theorem 1.9. [1]. If G is a 2-connected graph with maximum degree $\Delta = 3$ and $t(G) \geq 7$ then G is not L -borderenergetic, where $t(G)$ the number of vertices of degree 3 in G .

Theorem 1.10. [10].

- 1) There are no non-complete Q -borderenergetic graph of order $n \leq 5$ and 7.
- 2) There are exactly two non-complete Q -borderenergetic of order 6.
- 3) There exist exactly fourteen non-complete Q -borderenergetic graphs of order 8.
- 4) There exist exactly sixteen non-complete Q -borderenergetic graphs of order 9.

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Appendix

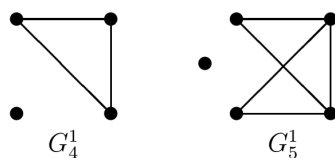


Figure 1. All Laplacian borderenergetic dis-connected graphs of order 4 and 5.

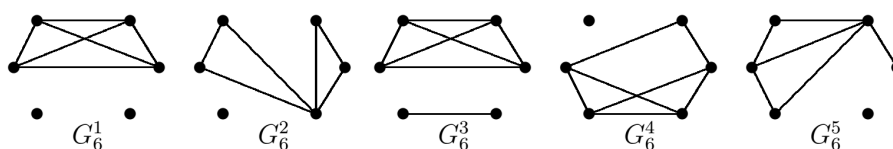


Figure 2. All Laplacian borderenergetic dis-connected graphs of order 6.

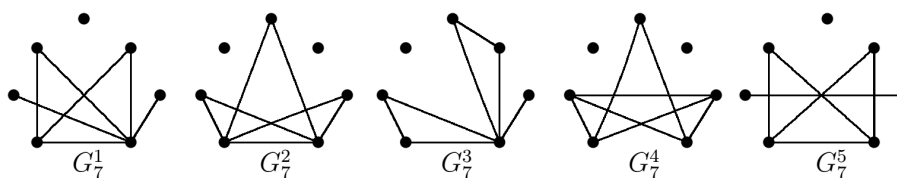


Figure 3. All Laplacian borderenergetic dis-connected graphs of order 7.

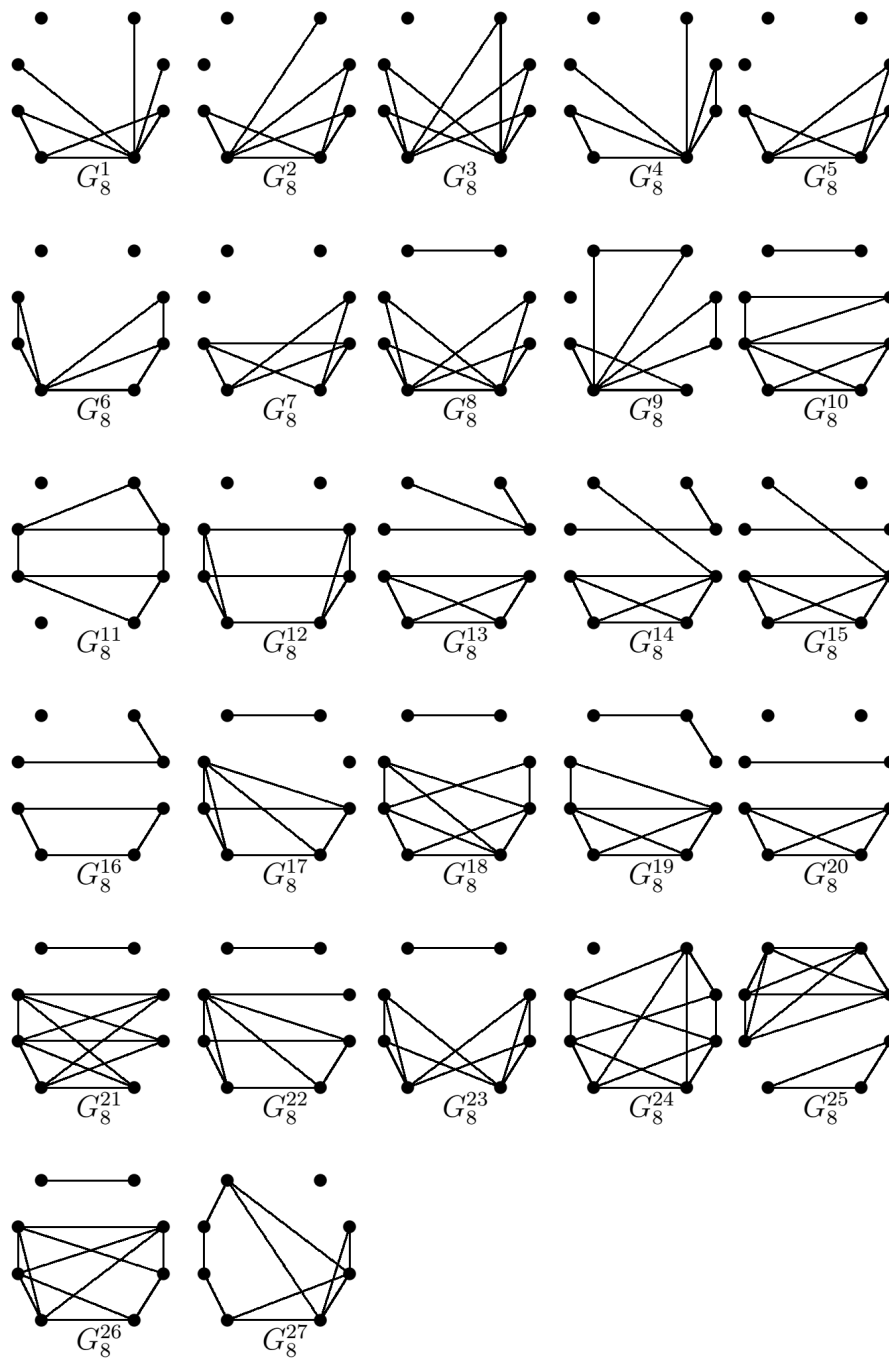


Figure 4. All Laplacian borderenergetic dis-connected graphs of order 8.

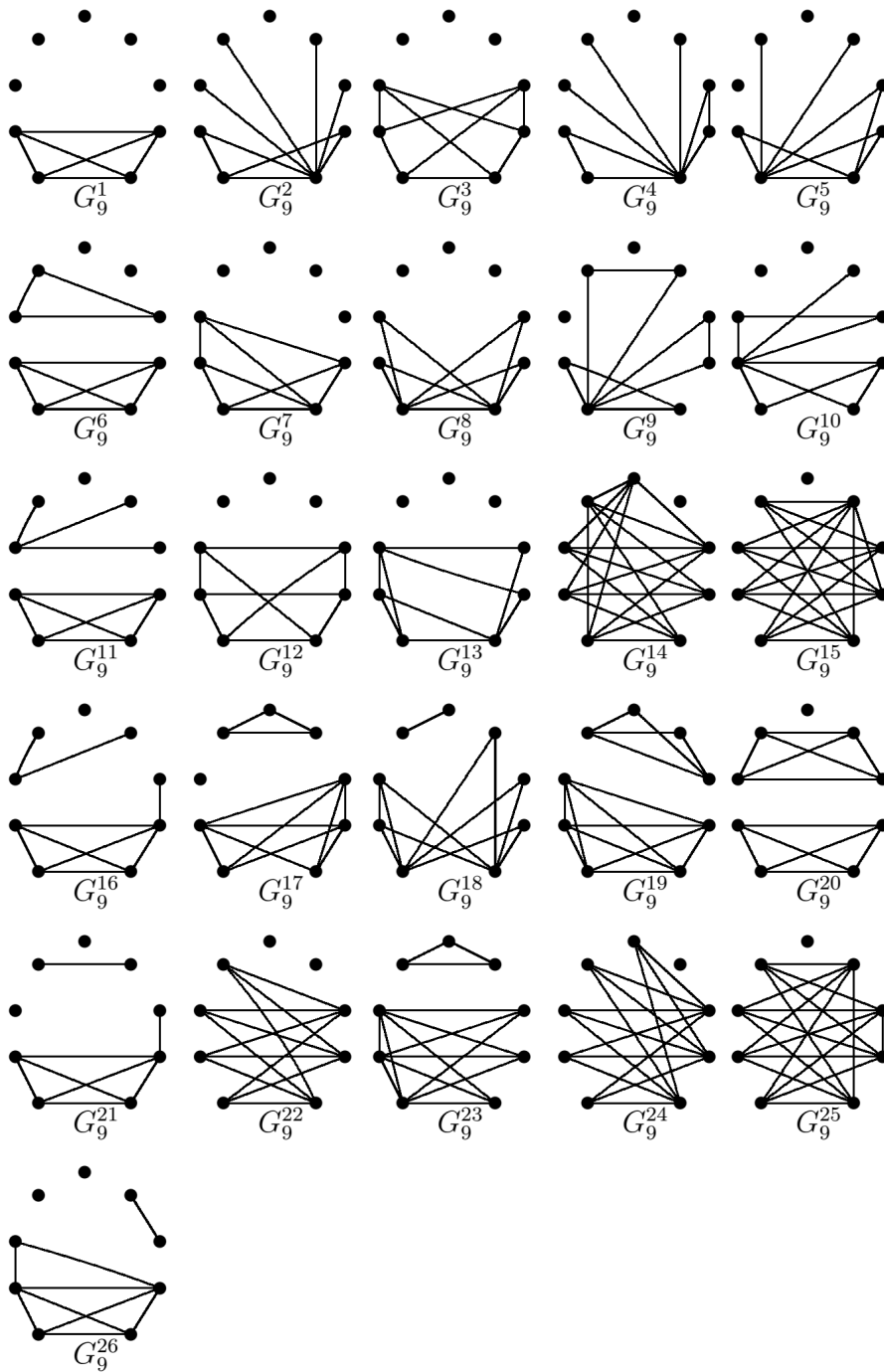


Figure 5. All Laplacian borderenergetic dis-connected graphs of order 9.