On the automorphism group of cubic polyhedral graphs

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Abstract. In the present paper, we introduce the automorphism group of cubic polyhedral graphs whose faces are triangles, quadrangles, pentagons and hexagons.

Keywords. polyhedral graph, automorphism group, fullerene.

1 Introduction

Carbon atoms can bond into very large molecules. Named, after U.S. engineer Buckminster Fuller (18951983), fullerenes are carbon molecules that have the same symmetry as a soccer ball, as shown in Figure 1. They are popularly called buckyballs. The most important member of fullerene graphs is C_{60} fullerene with exactly 60 carbon atoms. In general, a fullerene is a cubic planar graph having all faces 5- or 6-cycles, see Figure 2. Examples include the 20-vertex dodecahedral graph, 24-vertex generalized Petersen graph GP(12,2) and graph on 26 vertices truncated icosahedral graph.

A classical fullerene or briefly a fullerene is a cubic three connected graph whose faces entirely composed of pentagons and hexagones and we denote it by a PH-fullerene, see [18, 19]. The non-classical fullerenes are composed of triangles and hexagones or quadrangles and hexagones and we denote them by TH-fullerene or SH-fullerene, respectively. For see some problems concerning with fullerene graphs and many properties of them are studied in [1,2,4,7–9] as well as [11,13,16,17,20]. Fullerenes are special cases of a larger class of graphs,

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Figure 1. Fullerene C_{60}.

namely polyhedral graphs. A polyhedral graph is a three connected simple planar graph. In this paper, we consider only the cubic polyhedral graphs whose faces are a combination of triangles, quadrangles, pentagons and hexagones, see [4, 6].

An automorphism of graph $X = (V, E)$ is a bijection β on V which preserves the edge set E. In other words, $e = uv$ is an edge of E if and only if $e^\beta = u^\beta v^\beta$ is an edge of E. Here, the image of vertex u is denoted by u^β. The set of all automorphisms of graph X with the operation of composition is a group on $V(X)$ denoted by $Aut(X)$. Frucht [12] was the first who dealt with graph automorphism. Also, quantitative measures based on graph automorphism have been developed, see [3].

Cubic polyhedral graph with t triangular, s quadrilateral, p pentagonal and h hexagonal faces is denoted by a $(t, s, p, h)-$polyhedral or briefly a (t, s, p)-polyhedral graph. By these notations, a SPH-polyhedral graph is a planar graph whose faces are quadrangles, pentagons and hexagons. Let m be the number of edges in a given SPH-polyhedral graph F. In [11] Fowler and his co-authors showed that fullerenes are realizable within 28 point groups. In [21] Kutnar et al. proved that for any PH-fullerene graph F, $|Aut(F)|$ divides 120. The present authors in [14] proved that for given TH-fullerene F, $|Aut(F)|$ divides 24 and in [15] they proved that for given SH-polyhedral graph F, $|Aut(F)|$ divides 48. These results are given in the following theorem.

Theorem 1.1. *We have*

- *the size of automorphism group of classical fullerenes divides* 120 *[21].*

- *the size of automorphism group of TSH-fullerenes divides* 24, *[15].*

- *the size of automorphism group of SPH-fullerenes divides* 48, *[21].*

A TPH-polyhedral graph F is one whose faces are triangles, pentagons and hexagons. In this paper, we prove the following theorem.

Theorem A. *Let F be a TPH-polyhedral graph. Then the automorphism group of F is a subgroup of a $\{2, 3, 5\}$-group. Moreover, the order of $Aut(F)$ divides $2^2 \times 3$.*
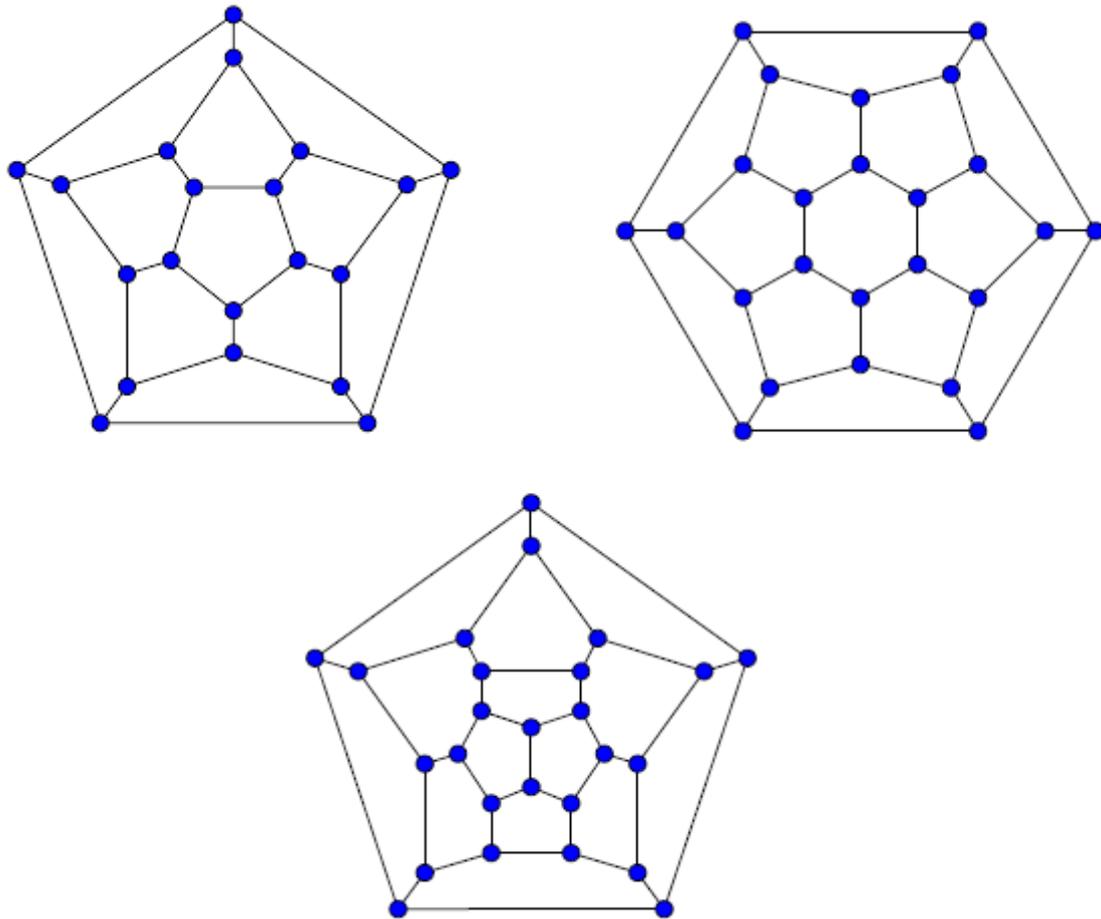
Figure 2. Planar graphs of Fullerenes C_{20}, C_{24} and C_{26}.

2 Definitions and Preliminaries

Let G be a group and Ω a non-empty set. An action of G on Ω denoted by $(G|\Omega)$ induces a group homomorphism φ from G into the symmetric group S_Ω on Ω, where $\varphi(g)^\alpha = g^\alpha$ $(\alpha \in \Omega)$. The orbit of an element $\alpha \in \Omega$ is denoted by α^G and it is defined as the set of all α^g where $g \in G$. Size of Ω is called the degree of this action. The kernel of this action is denoted by $Ker\varphi$. An action is faithful if $Ker\varphi = \{1\}$. The stabilizer of element $\alpha \in \Omega$ is defined as $G_\alpha = \{g \in G | \alpha^g = \alpha\}$. Let $H = G_\alpha$ then for $\alpha, \beta \in \Omega$ $(\alpha \neq \beta)$, H_β is denoted by $G_{\alpha,\beta}$. The orbit-stabilizer theorem implies that $|\alpha^G|.|G_\alpha| = |G|$. For every $g \in G$, let $fix(g) = \{\alpha \in X, \alpha^g = \alpha\}$, then we have:

Lemma 2.1. (Cauchy–Frobenius Lemma) *Let G acts on set Ω, then the number of orbits of G is*

$$\frac{1}{|G|} \sum_{g \in G} |fix(g)|.$$

Example 2.2. Consider the fullerene graph F_{96} depicted in Figure 3. If α denotes the rotation of F_{96} through an angle of $60°$ around an axis through the midpoints of the front and back

faces, then the corresponding permutation is $\alpha = (1,2,3,4,5,6)(7,10,14,17,20,24)(8,11,15,18,$
$21,25)$ $(9,12,16,1923,26)$ $(13,50,58,74,66,42)$ $(22,71,47,39,55,63)$ $(27,28,29,30,31,32)$ $(33,48,$
$56,72,64,40)$ $(34,49,57,73,65,41)(35,51,59,75,67,43)$ $(36,52,60,76,68,44)(37,53,61,77,69,45)$
$(38,54,62,78,70,46)$ $(79,80,81,82,83,84)(85,86,87,88,89,90)$ $(91,96,95,94,93,92)$. Thus, one
of orbits of subgroup $\langle \alpha \rangle$ containing the vertex 1 is $1^{\langle \alpha \rangle} = \{1,2,3,4,5,6\}$. Now, consider the
axis symmetry element which fixes vertices $\{1,4,8,18,43,44,59,60,85,88, 92,95\}$, the corre-
sponding permutation is $\beta = (2,6)$ $(3,5)$ $(7,9)$ $(10,26)$ $(11,25)(12,24)(13,71)$ $(14,23)$ $(15,21)$
$(16,20)$ $(17,19)(22,50)(27,28)(29,32)(30,31)$ $(33,70)$ $(34,69)$ $(35,67)$ $(36,68)$ $(37,65)$ $(38,64)$
$(39,66)$ $(40,46)$ $(41,45)$ $(42,47)$ $(48,78)(49,77)(51,75)$ $(52,76)$ $(53,73)$ $(54,72)$ $(55,74)(56,62)$
$(57,61)(58,63)(79,80)(81,84)(82,83)(86,90)(87,89)(91,93)(94,96)$.

Let $G = Aut(F_{96})$, clearly $G \geq \langle \alpha, \beta \rangle$ and the orbit-stabilizer property implies that $|G| =$
$|1^G| \times |G_1|$. Any symmetry of the polyhedral graph F_{96} which fixes vertex 1 must also fixes
the opposite vertex 4. By applying orbit-stabilizer property, we found that $|G_1| = |4^{G_1}| \times$
$|G_{1,4}|$. It is easy to prove that $|G_{1,4}| = 2$ and hence $|G| = 12$. On the other hand, $|\langle \alpha, \beta \rangle| = 12$,
where $\alpha^4 = \beta^2 = 1, \beta^{-1}\alpha\beta = \alpha^{-1}$. This leads us to conclude that $G = \langle \alpha, \beta \rangle \cong D_{12}$.
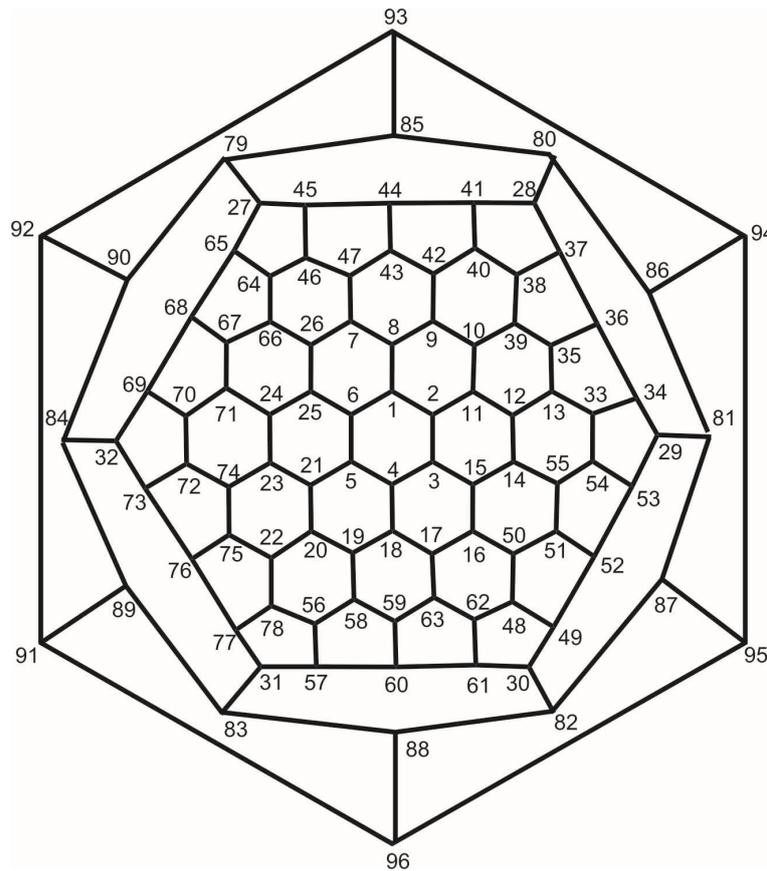

Figure 3. Labeling of fullerene F_{96}.

Example 2.3. Here, we compute the order of automorphism group of polyhedral graph F_{48}

depicted in Figure 4. Similar to the last example, if α denotes the rotation of F_{48} through an angle of $90°$ around an axis through the midpoints of the front and back faces, then the corresponding permutation is $\alpha = (1,3,5,7)\ (2,4,6,8)(9,15,26,21)\ (10,16,27,32)\ (11,17,28,22)(12,$ $29,23)(13,19,30,24)(14,20,31,25)(33,45,41,37)(34,46,42,38)(35,47,43,39)(36,48,44,40)$. Thus $1^{\langle\alpha\rangle} = \{1,3,5,7\}$ and consider the axis symmetry element which fixes no vertices: $\beta = (1,2)(3,8)(4,7)(5,6)(9,20)(10,19)(11,18)(12,17)(13,16)(14,15)(21,31)(22,29)(23,28)(24,$ $27)(25,26)(30,32)(33,40)(34,39)(35,38)(36,37)(41,48)(42,47)(43,46)(44,45)$.

If $G = Aut(F_{48})$, then $|G| = |2^G| \times |G_2|$ while no element fixes 2. This means that $|G_2| = 1$ and so $|G| = |2^G|$. It is clear that $2^{\langle\alpha,\beta\rangle} = \{1,2,3,4,5,6,7,8\}$ and thus $|2^{\langle\alpha,\beta\rangle}| = 8$. Similar to the last example, we can see that $|\langle\alpha,\beta\rangle| = 8$, where $\alpha^4 = \beta^2 = 1$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$, hence $Aut(F_{48})$ is isomorphic with dihedral group D_8.
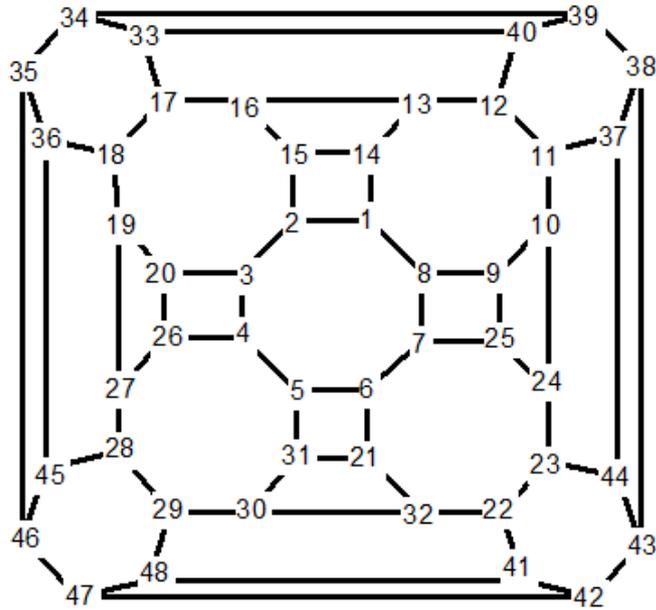

Figure 4. Labeling of fullerene F_{48}.

3 Main Results

Lemma 3.1. *Let F be a TPH-polyhedral graph with automorphism group $Aut(F)$. Then the stabilizer $Aut(F)_{(u,v,w)}$ of 2-arc (u,v,w) is trivial.*

Proof. It is similar to the proof of [21, Lemma 2]. $\qquad\square$

Proposition 3.2. *Let F be a TPH-polyhedral graph with automorphism group $Aut(F)$ and $u \in V(F)$. Then the stabilizer $Aut(F)_u$ of u is trivial or it is isomorphic to one of three groups: the cyclic group $\mathbb{Z}_2$, the cyclic group $\mathbb{Z}_3$ and the symmetric group S_3.*

Proof. It is similar to the proof of [21, Lemma 2]. $\qquad\square$

Proof of Theorem A. Let F be a TPH-polyhedral graph with a non-trivial automorphism group, $\mathcal{T}(F)$ be the set of triangles of F and $\mathcal{P}(F)$ be the set of pentagons of F. Let $A = Aut(F)$ and t, p be the number of triangles and pentagons, respectively. We can see that

$$|G| = |K_O| \times |(G/K_O)_T| \times |O| = 2^\alpha.3^\beta.|O|,$$

and so

$$|G| = 2^\alpha.3^\beta.5^\gamma.|O'|.$$

We distinguish the following cases:

Case 1. $t = 1$ and $p = 9$. We claim that $|Aut(F)|$ divides 3×2. Suppose 2^2 divides $|A|$ and $Syl_2(A)$ is of order 2^2. The order of orbits of $\mathcal{T}(F)$ is 1. By orbit- stabilizer theorem, we have $|K_T| = 2^2$, which is a contradiction. Let $|Syl_5(A)| = 5$, then $|K_T| = 5$, is a contradiction. If $|Syl_3(A)| = 3^2$, then we have $|K_T| = 3^2$, a contradiction.

Case 2. $t = 2$ and $p = 6$, we show that $|Aut(F)|$ divides 3×2^2. Suppose 2^3 divides $|A|$ and $Syl_2(A)$ is of order 2^3. Let $Syl_2(A)$ acts on triangles, then for an orbit O of this action, we have $|O| = 1$ or 2. By orbit-stabilizer theorem, if $|O| = 1$, then $|K_T| = 2^3$, is a contradiction and if $|O| = 2$, then $|K_T| = 2^2$, is a contradiction. Let $|Syl_5(A)| = 5$, then $|K_T| = 5$, is a contradiction. If $|Syl_3(A)| = 3^2$, then we have $|O| = 1$ and so $|K_T| = 3^2$, is a contradiction.

Case 3. $t = 3$ and $p = 3$, we prove that $|Aut(F)|$ divides 3×2^2. Let 2^3 divides $|A|$ and $Syl_2(A)$ be of order 2^3 acting on the set of triangles $\mathcal{T}$. Hence, $|O| = 1$ or 2, similar to the last discussion, both of them are contradictions. Also, $|Syl_5(A)| = 5$ is a contradiction. If $|Syl_3(A)| = 3^2$, then the orbits of the action $Syl_3(A)$ on $\mathcal{P}(F)$ are of order 3 and so $|K_P| = 3$, is a contradiction.

It should be noted that in a given polyhedral F, no two triangles are adjacent, since F is three connected.

Theorem 3.3. *Let F be a TSH-polyhedral graph. Then $Aut(F)$ is a subgroup of a $\{2,3\}$-group. Moreover, the order of $Aut(F)$ divides* 24.

Proof. By using Euler's theorem, if $s = 0$, then $t = 4$ and F is TH- fullerene. On the other hand, if $s = 6$, then $t = 0$ and F is a SH-fullerene. Let F be a TSH-polyhedral graph with at least one triangle and one square. We show that $|Aut(F)|$ divides 24. First, we prove that $Syl_2(F)$ is of order 8. Suppose on the contrary that 2^4 divides $|A|$ and $Syl_2(A)$ is of order 2^4. Let $Syl_2(A)$ acts on the set of triangles. Clearly the order of an orbit of this action is 1 or 2. By orbit-stabilizer theorem, if $|O| = 1$, then $|K_T| = 2^3$, which is a contradiction; and if $|O| = 2$, then $|K_T| = 2^2$, which is a contradiction. Also, $|Syl_5(A)| = 5$ yields a contradiction. If $|Syl_3(A)| = 3^2$, $|K_q| = 3^2$ or $|K_q| = 3$, then we have a contradiction. This completes the proof. $\square$

In [6] the authors derived the list of allowed symmetry groups for each class where they constructed the smallest polyhedron for each allowed symmetry. In other words, we have two following theorems, see [5, 10, 11].

Theorem 3.4. *For the bifaced cubic polyhedra described by the triple (t, s, p), the possible point groups and vertex counts of minimal examples are*

  i. $(t, s, p) = (4, 0, 0)$:
      $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, A_4, S_4$.

  ii. $(t, s, p) = (0, 6, 0)$:
      $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$
      $\mathbb{Z}_2 \times S_3, D_{12}, D_6, \mathbb{Z}_2 \times D_{12}, \mathbb{Z}_2 \times D_6, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_6$.

  iii. $(t, s, p) = (0, 0, 12)$:
      $C_1, \mathbb{Z}_2, A_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, S_6, S_3, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_2 \times \mathbb{Z}_5, D_{12}, \mathbb{Z}_2 \times S_3,$
      $A_4, D_{20}, \mathbb{Z}_2 \ltimes D_{12}, D_{24}, S_4, A_4 \times \mathbb{Z}_2, A_5, \mathbb{Z}_2 \times A_5$.

Theorem 3.5. *For cubic polyhedra with at least two face sizes chosen from $\{3, 4, 5\}$ and no face of size greater than 6 described by the triple (t, s, p), the possible point groups and vertex counts of minimal examples are*

  i. $(t, s, p) = (3, 1, 1)$:
      $C_1, \mathbb{Z}_2$.

  ii. $(t, s, p) = (3, 0, 3)$:
      $C_1, \mathbb{Z}_2, A_3, S_3, \mathbb{Z}_2 \times \mathbb{Z}_3$.

  iii. $(t, s, p) = (2, 3, 0)$:
      $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_{12}$.

  iv. $(t, s, p) = (2, 2, 2)$:
      $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$.

  v. $(t, s, p) = (2, 1, 4)$:
      $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$.

  vi. $(t, s, p) = (2, 0, 6)$:
      $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, \mathbb{Z}_2 \times S_3, D_{12}$.

  vii. $(t, s, p) = (1, 4, 1)$:
      $C_1, \mathbb{Z}_2$.

  viii. $(t, s, p) = (1, 3, 3)$:
      $C_1, \mathbb{Z}_2, A_3, S_3$.

ix. $(t,s,p) = (1,2,5)$:
   $C_1, \mathbb{Z}_2$.

x. $(t,s,p) = (1,1,7)$:
   $C_1, \mathbb{Z}_2$.

xi. $(t,s,p) = (1,0,9)$:
   $C_1, \mathbb{Z}_2, A_3, S_3$.

xii. $(t,s,p) = (0,5,2)$:
   $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_5, D_{20}$.

xiii. $(t,s,p) = (0,4,4)$:
   $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_4$.

xiv. $(t,s,p) = (0,3,6)$:
   $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, A_3, \mathbb{Z}_2 \times \mathbb{Z}_3, S_3, D_{12}$.

xv. $(t,s,p) = (0,2,8)$:
   $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_2 \times D_8, D_{16}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$.

xvi. $(t,s,p) = (0,1,10)$:
   $C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$.

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