VARIATIONAL DISCRETIZATION AND MIXED METHODS FOR SEMILINEAR PARABOLIC OPTIMAL CONTROL PROBLEMS WITH INTEGRAL CONSTRAINT

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ABSTRACT
The aim of this work is to investigate the variational discretization and mixed finite element methods for optimal control problem governed by semi linear parabolic equations with integral constraint. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is not discreted. Optimal error estimates in L^2 are established for the state and the control variable. As a result, it can be proved that the discrete solutions possess the convergence property of order h. Finally, a numerical example is presented which confirms the theoretical results.

KEYWORDS: Priori error estimates, Parabolic optimal control, Integral constraint, Mixed finite element method, Variational discretization

INTRODUCTION
The finite element method of optimal control problems plays an important role in numerical methods [1-2]. Systematic introduction of the finite element method for optimal control problems can be found in [3].
In many control problems, the objective functional contains gradients of the state variables. Thus the accuracy of the gradient is important in numerical approximation of the state equations. In finite element method, mixed finite element methods are widely used to approximate flux variables, although there is only very limited research work on analyzing such elements for optimal control problems. More recently, some preliminary work have been done on a posteriori error estimates, error estimates of mixed finite element methods for optimal control problems [4-7]. In [8], the author presents the variational discretization concept for optimal control problems with control constraints. However, it doesn't seem to be straightforward to extend these existing techniques to the semilinear parabolic optimal control problems.

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In [9], the mixed finite element methods were used to discrete nonlinear elliptic optimal control problems and derive an L^2 priori error estimates. An L^2 and L^∞ priori error estimates for linear parabolic optimal control problems have been obtained when the space discretization of the state variable is done using usual mixed finite elements, the time discretization is based on different methods, and the control is approximated by piecewise constant elements in [10]. In this paper variational discretization and semi-discrete mixed finite element methods studied for semilinear parabolic optimal control problems with integral constraint. The state and co-state are approximated by the lowest order Raviart-Thomas mixed finite element spaces and the control is not discreted. An L^2 optimal error estimate for the state and control variable is structured and is proved that all the discrete variables possess the convergence property of order h.
The following general semilinear parabolic optimal control problem is considered:

$$\min_{u\in K\subset U}\left\{\int_0^T \left(g_1(\vec{p})+g_2(y)+j(u)\right)dt\right\}$$
$$y_t(x,t)+div\vec{p}(x,t)+\phi(y(x,t))=f(x,t)+u(x,t),\quad x\in\Omega,$$
$$\vec{p}(x,t)=-A(x)\nabla y(x,t),\quad x\in\Omega,$$
$$y(x,t)=0, x\in\partial\Omega, t\in J,\ y(x,0)=y_0(x),\quad x\in\Omega,$$

where the bounded open set $\Omega\in R^2$ is a convex polygon with the boundary $\partial\Omega$, $J=[0,T]$, $\vec{p}$, y are the state variables, u is the control variable. It is assumed that $f\in L^2(J;L^2(\Omega))$, g_1, g_2, and j are differentiable on $L^2(\Omega)^2$, $L^2(\Omega)$, $L^2(\Omega)$, respectively.
For any $r>0$ the function $\phi(y)\in W^2(-r,r)$, $\phi'(y)\in L^2(\Omega)$ for any $y\in H^1(\Omega)$, and $\phi'(y)\geq 0$. Furthermore, it is assumed that coefficient matrix $A(x)=\left(a_{ij}(x)\right)_{2\times 2}\in L(\Omega,R^{2\times 2})$ is a symmetric 2×2 matrix and there is a constant $c>0$ satisfying for any vector $X\in R^2$,

$$X^TAX\geq c\|X\|^2_{R^2}.$$

Here, the admissible set of the control variable K is defined by:

$$K=\left\{u\in L^2(J;L^2(\Omega)):\int_\Omega udx\geq 0\right\}.$$

The plan of this paper is as follows. In section II, the variational discretization and mixed finite element methods are structured for optimal control problems governed by semilinear parabolic equations with integral constraint. In section III, a priori error estimates for the variational discretization and mixed finite element approximation is derived for the optimal control problems. A numerical example is presented in section IV.

VARIATIONAL DISCRETIZATION AND MIXED METHODS

The variational discretization and mixed finite element discretization of semilinear parabolic optimal control problems with integral constraint are described below. Let $W=L^2(\Omega)$ and:

$$\vec{V}=H(div;\Omega)=\{\vec{v}\in L^2(\Omega)^2, div\,\vec{v}\in L^2(\Omega)\}$$

endowed with the norm given by:

$$\|\vec{v}\|_{H(div;\Omega)}=\left(\|\vec{v}\|^2_{0,\Omega}+\|div\,\vec{v}\|^2_{0,\Omega}\right)^{1/2}$$

The original optimal control problem is recast as the following weak form: find $(\vec{p},y,u)\in\vec{V}\times W\times K$ such that:

$$\min_{u\in K\subset U}\left\{\int_0^T \big(g_1(\vec{p})+g_2(y)+j(u)\big)dt\right\}$$
$$(A^{-1}\vec{p},\vec{v})-(y,div\vec{v})=0, \qquad \forall\vec{v}\in\vec{V},$$
$$(y_t,w)+(div\vec{p},w)+(\phi(y),w)=(f+u,w), \quad \forall w\in W,$$
$$y(x,0)=y_0(x), \qquad \forall x\in\Omega$$

It is well known that the above optimal control problem has a solution $(\vec{p},y,u)$, and that a triplet $(\vec{p},y,u)$ is the solution of the above optimal control problem if and only if there is a co-state $(\vec{q},z)\in\vec{V}\times W$ such that $(\vec{p},y,\vec{q},z,u)$ satisfies the following optimality conditions (1):

$$(A^{-1}\vec{p},\vec{v})-(y,div\vec{v})=0, \qquad \forall\vec{v}\in\vec{V},$$
$$(y_t,w)+(div\,\vec{p},w)+(\phi(y),w)=(f+u,w), \quad \forall w\in W,$$
$$y(x,0)=y_0(x), \qquad \forall x\in\Omega,$$
$$(A^{-1}\vec{q},\vec{v})-(z,div\vec{v})=(g_1^{'}(\vec{p}),\vec{v}), \qquad \forall\vec{v}\in\vec{V},$$
$$(z_t,w)+(div\vec{q},w)+(\phi^{'}(y)z,w)=(g_2^{'}(y),w), \quad \forall w\in W,$$
$$z(x,T)=0, \qquad \forall x\in\Omega,$$
$$\int_0^T (z+j^{'}(u),\tilde{u}-u)_U\,dt\ge 0, \qquad \forall\tilde{u}\in K,$$

where $(\cdot,\cdot)_U$ is the inner product of U. For simplicity, the product $(\cdot,\cdot)_U$ will be denoted as $(\cdot,\cdot)$.

Let Γ_h be regular triangulation of Ω. These are assumed to satisfy the angle condition which means that there is a positive constant C independent of h such that for all $T\in\Gamma_h$, $C^{-1}h_T^2\le|T|\le Ch_T^2$, where $|T|$ is the area of T and h_T is the diameter of T. Let $h=\max(h_T)$.

Let $V_h\times W_h\subset V\times W$ denote the Raviart-Thomas spaces of the lowest order associated with the triangulation Γ_h of Ω. P_k denotes the space of polynomials of total degree maximally k. If T is a triangle then:

$$\vec{V}(T)=\{v\in P_0^2(T)+x\cdot P_0(T)\}.$$
$$\vec{V}_h=\{v_h\in\vec{V}:\forall T\in\Gamma_h, v_h|_T\in\vec{V}(T)\},$$
$$W_h=\{w_h\in W:\forall T\in\Gamma_h, w_h|_T\in P_0(T)\}$$

By the definition of finite element subspace, the mixed finite element approximation of optimal control problem is as follows:

compute $(\vec{p}_h,y_h,u_h)\in\vec{V}_h\times W_h\times K$ such that:

$$\min_{u_h\in K}\left\{\int_0^T \big(g_1(\vec{p}_h)+g_2(y_h)+j(u_h)\big)dt\right\}$$
$$(A^{-1}\vec{p}_h,\vec{v}_h)-(y_h,div\vec{v}_h)=0, \qquad \forall\vec{v}_h\in\vec{V}_h,$$
$$(y_{ht},w_h)+(div\vec{p}_h,w_h)+(\phi(y_h),w_h)=(f+u_h,w_h), \quad \forall w_h\in W_h,$$
$$y(x,0)=y_0(x), \qquad \forall x\in\Omega$$

where $y_0^h(x)\in W_h$ is an approximation of y_0.

It is well known that the above optimal control problem has a solution $(\vec{p}_h, y_h, u_h)$, and that a triplet $(\vec{p}_h, y_h, u_h)$ is the solution of the above optimal control problem if and only if there is a co-state $(\vec{q}_h, z_h) \in \vec{V}_h \times W_h$ such that $(\vec{p}_h, y_h, \vec{q}_h, z_h, u_h)$ satisfies the following optimality conditions (2) :

$$
\begin{aligned}
&(A^{-1}\vec{p}_h, \vec{v}_h) - (y_h, div\vec{w}_h) = 0, && \forall \vec{v}_h \in \vec{V}_h,\\
&(y_{ht}, w_h) + (div\vec{p}_h, w_h) + (\phi(y_h), w_h) = (f + u_h, w_h), && \forall w_h \in W_h,\\
&y_h(x,0) = y_0(x), && \forall x \in \Omega,\\
&(A^{-1}\vec{q}_h, \vec{v}_h) - (z_h, div\vec{w}_h) = (g_1'(\vec{p}_h), \vec{v}_h), && \forall \vec{v}_h \in \vec{V}_h,\\
&(z_{ht}, w_h) + (div\vec{q}_h, w_h) + (\phi'(y_h)z_h, w_h) = (g_2'(y_h), w_h), && \forall w_h \in W_h,\\
&z_h(x,T) = 0, && \forall x \in \Omega,\\
&\int_0^T (z_h + j'(u_h), \tilde{u}_h - u_h)_U dt \geq 0, && \forall \tilde{u}_h \in K,
\end{aligned}
$$

For $\varphi \in W_h$, it shall be written:

$$\phi(\varphi) - \phi(\rho) = -\tilde{\phi}'(\varphi)(\rho - \varphi) = -\phi'(\rho)(\rho - \varphi) + \tilde{\phi}''(\varphi)(\rho - \varphi)^2,$$

where:

$$\tilde{\phi}'(\varphi) = \int_0^1 \phi'(\varphi + t(\rho - \varphi))dt,$$

$$\tilde{\phi}''(\varphi) = \int_0^1 (1-t)\phi''(\rho + t(\varphi - \rho))dt$$

are bounded functions on $\overline{\Omega}$.

A PRIORI ERROR ESTIMATES

For any control function $\tilde{u} \in K$, the discrete state solution $(\vec{p}_h(\tilde{u}), y_h(\tilde{u}), \vec{q}_h(\tilde{u}), z_h(\tilde{u}))$ associated with $\tilde{u}$ satisfies (3):

$$
\begin{aligned}
&(A^{-1}\vec{p}_h(\tilde{u}), \vec{v}_h) - (y_h(\tilde{u}), div\,\vec{v}_h) = 0,\\
&(y_{ht}(\tilde{u}), w_h) + (div\ p_h(\tilde{u}), w_h) + (\phi(y_h(\tilde{u})), w_h) = (f + \tilde{u}, w_h),\\
&y_h(\tilde{u})(x,0) = y_0(x),\\
&(A^{-1}\vec{q}_h(\tilde{u}), \vec{v}_h) - (z_h(\tilde{u}), div\,\vec{v}_h) = (g_1'(\vec{p}_h(\tilde{u})), \vec{v}_h),\\
&-(z_{ht}(\tilde{u}), w_h) + (div\,\vec{q}_h(\tilde{u}), w_h) + (\phi'(y_h(\tilde{u}))z_h(\tilde{u}), w_h) = (g_2'(y_h(\tilde{u})), w_h),\\
&z_h(\tilde{u})(x,T) = 0,
\end{aligned}
$$

for any $\vec{v}_h \in \vec{V}_h, w_h \in W_h$.

Lemma 1 There is a positive constant C independent of h such that:

$$\|\vec{p} - \vec{p}_h(u)\|_{L^2(J;L^2(\Omega))} + \|y - y_h(u)\|_{L^\infty(J;L^2(\Omega))} \leq Ch,$$

$$\|\vec{q} - \vec{q}_h(u)\|_{L^2(J;L^2(\Omega))} + \|z - z_h(u)\|_{L^\infty(J;L^2(\Omega))} \leq Ch$$

By applying the intermediate errors, the errors can be decomposed as:

$$\varepsilon_1 = \vec{p}_h(u) - \vec{p}_h, \qquad r_1 = y_h(u) - y_h,$$

$$\varepsilon_2 = \vec{q}_h(u) - \vec{q}_h, \qquad r_2 = z_h(u) - z_h.$$

From (2) and (3):

$$(A^{-1}\varepsilon_1,\vec{v}_h)-(r_1,div\,\vec{v}_h)=0,$$
$$(r_{1t},w_h)+(div\,\varepsilon_1,w_h)+(\tilde{\phi}^{'}(y_h(u))r_1,w_h)=(u-u_h,w_h),$$
$$(A^{-1}\varepsilon_1,\vec{v}_h)-(r_1,div\,\vec{v}_h)=(g_1^{'}(\vec{p}_h(\tilde{u}))-g_1^{'}(\vec{p}_h),\vec{v}_h),$$
$$(r_{2t},w_h)+(div\,\varepsilon_2,w_h)+(\phi^{'}(y_h(u))r_2+\tilde{\phi}^{''}(y_h(u))r_1z_h,w_h)=(g_2^{'}(y_h(\tilde{u}))-g_2^{'}(y_h),w_h)$$

By Lemma 2.1 in [11], the following error estimates can be established.

Lemma 2 There is a positive constant C independent of h such that:

$$\|\vec{p}_h(u)-\vec{p}_h\|_{L^2(J;L^2(\Omega))}+\|y_h(u)-y_h\|_{L^\infty(J;L^2(\Omega))}\le C(u-u_h),$$
$$\|\vec{q}_h(u)-\vec{q}_h\|_{L^2(J;L^2(\Omega))}+\|z_h(u)-z_h\|_{L^\infty(J;L^2(\Omega))}\le C(u-u_h).$$

Let $(\vec{p}(u),y(u))$ and $(\vec{p}_h(u_h),y_h(u_h))$ be the solutions of (1) and (2), respectively. Let $J(\cdot):U\to R$ be a G-differential convex functional which satisfies the following form:

$$J(u)=g_1(\vec{p})+g_2(y)+j(u),$$
$$J_h(u_h)=g_1(\vec{p}_h)+g_2(y_h)+j(u_h).$$

It can be shown that:

$$(J^{'}(u),v)=(j^{'}(u)+z,v),$$
$$(J_h^{'}(u),v)=(j^{'}(u)+z_h(u),v),$$
$$(J_h^{'}(u_h),v)=(j^{'}(u_h)+z_h,v).$$

Then the results are obtained:

Theorem 1 Let $(\vec{p},y,\vec{q},z,u)\in(\vec{V}\times W)^2\times K$ and $(\vec{p}_h,y_h,\vec{q}_h,z_h,u_h)\in(\vec{V}_h\times W_h)^2\times K$ be solutions of (1) and (2) , respectively. It is assumed that $z+j^{'}(u)\in H^1(\Omega)$. Then:

$$\|u-u_h\|_{L^2(J;L^2(\Omega))}\le Ch,$$
$$\|\vec{p}-\vec{p}_h\|_{L^2(J;L^2(\Omega))}+\|y-y_h\|_{L^2(J;L^2(\Omega))}\le Ch,$$
$$\|\vec{q}-\vec{q}_h\|_{L^2(J;L^2(\Omega))}+\|z-z_h\|_{L^2(J;L^2(\Omega))}\le Ch.$$

NUMERICAL EXAMPLE

In this section, the priori error estimates are to be validated for the error in the control, state, and co-state. The discretization was already simplified: the control function u is not discretized, whereas the state $(\vec{p},y)$ and the co-state $(\vec{q},z)$ were approximations by the RT0 mixed finite element functions [12].

The numerical example is the following optimal control problem:

$$\min_{u\in K\subset U}\left\{\int_0^T \tfrac{1}{2}\|\vec{p}-\vec{p}_d\|^2+\tfrac{1}{2}\|y-y_d\|^2+\tfrac{1}{2}\|u\|^2\right\}$$
$$y_t+div\vec{p}+y^3=f+u,\qquad x\in\Omega,$$
$$\vec{p}=-\nabla y,\qquad x\in\Omega,$$
$$y(x,t)=0,x\in\partial\Omega,t\in J,\quad y(x,0)=0,\qquad x\in\Omega,$$
$$-z_t+div\vec{q}+3y^2z=y-y_d,\qquad x\in\Omega,$$
$$\vec{q}=-\nabla z-\vec{p}+\vec{p}_d,\qquad x\in\Omega,$$
$$z(x,t)=0,x\in\partial\Omega,t\in J,\quad z(x,T)=0,\qquad x\in\Omega$$

The admissible set of the control variable is:

$$K = \left\{ u \in L^2(J; L^2(\Omega)) : \int_\Omega u \geq 0 \right\}$$

Let:

$$\begin{aligned}
u &= \max(\bar{z}, 0) - z, \\
f &= y_t + div\,\vec{p} + y^3 - u \\
y_d &= y + z_t - 3y^2 z \\
y &= \sin \pi x_1 \sin \pi x_2 \sin \pi t \\
z &= \sin \pi x_1 \sin \pi x_2 \sin \pi t \\
\vec{q} &= \vec{p}_d = (0,0) \\
\vec{p} &= -\begin{pmatrix} \pi \cos \pi x_1 \sin \pi x_2 \sin \pi t \\ \pi \sin \pi x_1 \cos \pi x_2 \sin \pi t \end{pmatrix}.
\end{aligned}$$

The same mesh partition for the state and the control are adopted such that $\Delta t = h$ in the test. The solutions are computed on a series of uniform meshes. Fig. (1) shows surfaces of the approximation solution u_h at $t = 0.25$. The errors obtained on the variational discretization and mixed finite element approximation for state functions and control function are presented in Table (1). Furthermore, the convergence orders are shown by slopes in Fig. (2). This is consistent with the results previously proved.

Table 1. The numerical errors for state and control functions.

h	Errors				
	u	$\vec{p}$	y	$\vec{q}$	z
1/16	2.64e-02	6.15e-01	3.18e-02	2.63e-02	3.18e-02
1/32	1.32e-02	3.05e-01	1.56e-02	1.31e-02	1.56e-02
1/64	6.62e-03	1.53e-01	7.75e-03	6.61e-03	7.75e-03
1/128	3.31e-03	7.51e-02	3.88e-03	3.30e-03	3.87e-03

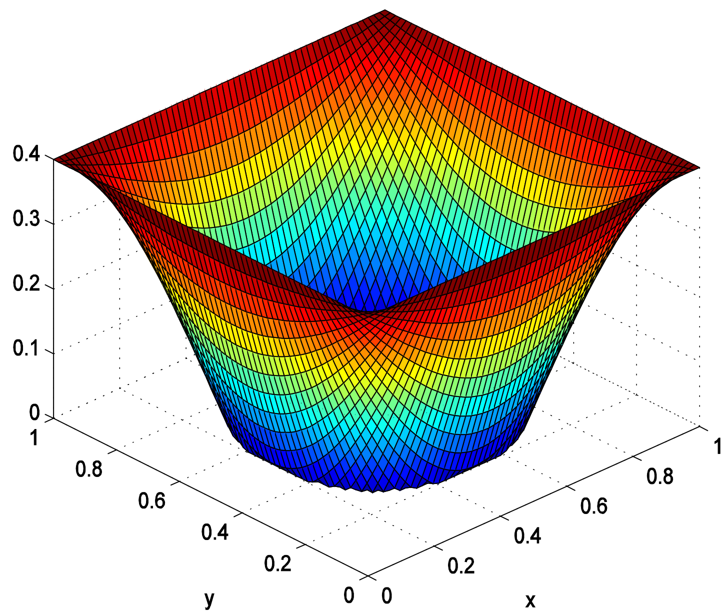

Fig. 1. The profile of the control solution at $t = 0.25$.

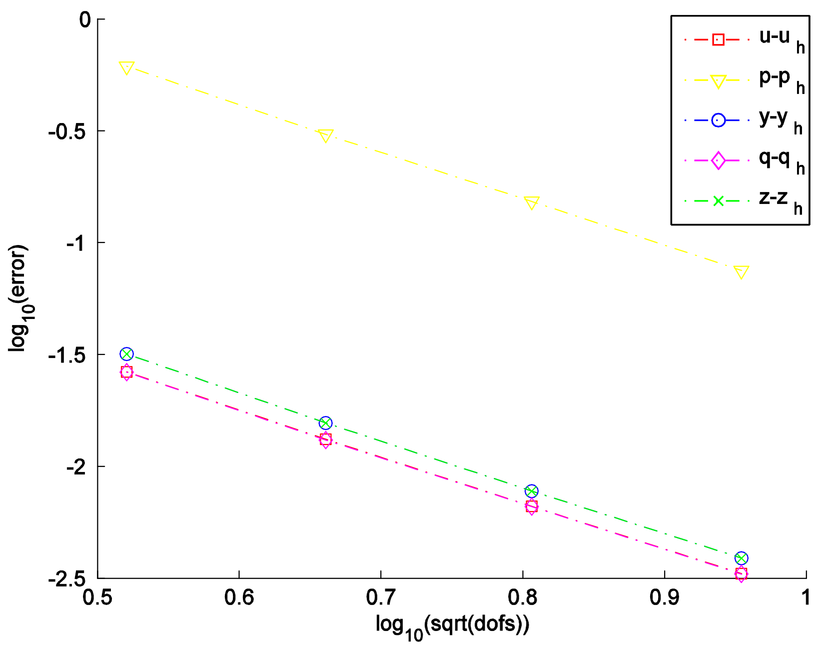

Fig. 2. The convergence orders on triangle mesh grids.

CONCLUSIONS

In this paper, a priori error estimates is investigated for variational discretization and mixed finite element methods of the semilinear parabolic optimal control problems with integral constraint. The state and the co-state are approximated by the mixed finite element spaces and the control is not discreted.

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